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On the Solution of Jordan's System of Difference Equations

ABSTRACT. By means of generalized Bernoulli numbers an explicit solution is given for the homogeneous system of difference equations in Jordan's normal form.

KEY WORDS. Difference equations, Jordan's normal form, Bernoulli numbers.

1 Introduction

In U. Krause and T. Nesemann [1], Lemma 3.9, there was determined the general solution of the system

$$x_i(t+1) = \lambda x_i(t) + x_{i+1}(t) \tag{1.1}$$

for $1 \leq i \leq n$, $t \in \mathbb{N}$ and $x_{n+1}(t) \equiv 0$. For $\lambda \neq 0$ it reads

$$x_i(t) = \lambda^{t+i-1} \Delta^{i-1} p(t)\,, \tag{1.2}$$

where p is an arbitrary polynomial of degree $\leq n-1$ and $\Delta p(t) = p(t+1) - p(t)$. The system (1.1) corresponds to a homogeneous matrix difference equation in Jordan's normal form with upper unities. After simple modifications, the result can easily be transferred to the system

$$y_i(t+1) = \lambda y_i(t) + y_{i-1}(t) \tag{1.3}$$

for $0 \leq i \leq n$ with $y_{-1}(t) \equiv 0$, where the solution turns over into

$$y_i(t) = \lambda^{t-i} \Delta^{n-i} p(t) \tag{1.4}$$

with an arbitrary polynomial p of degree $\leq n$. The system (1.3) corresponds to Jordan's normal form with lower unities.

The solution (1.4) has the disadvantage that its shape depends on n. In the sequel we derive a representation for the solution of (1.3) which is independent of n and in which, surprisingly, the Bernoulli numbers appear. The result can be transferred to inhomogeneous equations.

Let us mention that (1.2) and (1.4) are also the general solutions of (1.1) and (1.3), respectively, in the case $t \in \mathbb{R}$, if p denotes a polynomial in t with arbitrary 1-periodic coefficients. An analogous remark comes true for the latter solutions.

2 Solution of (1.3)

In order to solve (1.3) with $y_{-1}(t) \equiv 0$ and $\lambda \neq 0$ for $i = 0, 1, 2, \dots$ we make the ansatz

$$y_i(t) = \lambda^{t-i} \sum_{j=0}^{i} \frac{1}{j!} a_{i,i-j} t^j \,. \tag{2.1}$$

The homogeneous equation, corresponding to (1.3) with fixed i, has the general solution $y_i = c_i \lambda^i$ with arbitrary c_i so that in (1.3) a_{ii} must remain arbitrarily. Comparing

$$y_i(t+1) - \lambda y_i(t) = \lambda^{t+1-i} \sum_{j=1}^{i} \frac{1}{j!} a_{i,i-j} \sum_{k=0}^{j-1} \binom{j}{k} t^k$$

with (2.1) for $i-1$ instead of i, equation (1.3) implies

$$\sum_{j=k+1}^{i} \frac{1}{(j-k)!} a_{i,i-j} = a_{i-1,i-1-k}$$

for $k = 0, 1, \dots, i-1$ and therefore

$$\sum_{j=0}^{k} \frac{1}{(i+1-j)!} a_{ij} = a_{i-1,k} \tag{2.2}$$

for the same k. Defining auxiliary coefficients a_{ij} also for $j > i$, we introduce the generating functions

$$f_i(z) = \sum_{j=0}^{\infty} a_{ij} z^j \tag{2.3}$$

as formal power series. Choosing the new a_{ij} for $i = 0$ arbitrarily and for $i \in \mathbb{N}$ in a suitable way, the equations (2.2) are equivalent to the recursions

$$\frac{e^z - 1}{z} f_i(z) = f_{i-1}(z)$$

with the solution

$$f_i(z) = \left(\frac{z}{e^z - 1}\right)^i f_0(z)\,. \tag{2.4}$$

Now, we define coefficients b_{ij} by

$$\left(\frac{z}{e^z - 1}\right)^i = \sum_{j=0}^{\infty} b_{ij} z^j \tag{2.5}$$

so that (2.3)-(2.4) imply the relations

$$a_{ij} = \sum_{k=0}^{j} b_{ik}\, a_{0,j-k}\,, \tag{2.6}$$

which, in view of (2.2), are valid for $0 \le j \le i-1$. But we can use (2.6) also for $j = i$, if we take a_{0i} as arbitrary constants instead of a_{ii}. The foregoing considerations can be summarized as follows:

Proposition *For $\lambda \neq 0$ the equations (1.3) with $i \in \mathbb{N}_0$ and $y_{-1}(t) \equiv 0$ have the general solution (2.1) with (2.6), the coefficients b_{ik} defined by (2.5), and arbitrary constants a_{0j}.*

Usually, the coefficients in (2.5) are written in the form

$$b_{ij} = \frac{1}{j!} B_j^{(i)}\,,$$

where $B_j^{(i)}$ are the generalized Bernoulli numbers, cf. [2], p. 145 or [3], p. 4. The generalized Bernoulli numbers are polynomials in i of degree j. For $0 \le j \le 12$ they are listed in [2], p. 459, from which we obtain in particular

$$b_{i0} = 1\,, \quad b_{i1} = -\frac{i}{2}\,, \quad b_{i2} = \frac{i}{24}(3i-1)\,, \quad b_{i3} = -\frac{i^2}{48}(i-1)\,.$$

These results can be checked by means of the recursions

$$b_{i+1,j} = \left(1 - \frac{j}{i}\right) b_{ij} - b_{i,j-1} \qquad (i, j \in \mathbb{N})\,,$$

cf. [2], p. 145, and the initial values $b_{i0} = 1$, $b_{1j} = \frac{1}{j!} B_j$ for $i, j \in \mathbb{N}_0$, where B_j are the ordinary Bernoulli numbers.

3 The inhomogeneous case

The foregoing considerations also allow to solve the inhomogeneous equation

$$y(t+1) = \lambda y(t) + t^{i-1} \lambda^t\,, \tag{3.1}$$

$i \in \mathbb{N}$, if we interprete (3.1) as equation (1.3) with $y(t) = y_i(t)$ and $t^{i-1}\lambda^t = y_{i-1}(t)$ with (2.1), i.e. in particular $a_{i-1,0} = (i-1)!\lambda^{i-1}$ and $a_{i-1,j} = 0$ elsewhere. Now, instead of (2.4) we only need the relation

$$f_i(z) = \frac{z}{e^z - 1} f_{i-1}(z)\,,$$

and since $f_{i-1}(z) = (i-1)!\lambda^{i-1}$, we obtain

$$a_{ij} = (i-1)!\lambda^{i-1}\frac{1}{j!}B_j \tag{3.2}$$

for $j = 0, 1, \ldots, i-1$, whereas a_{ii} is an arbitrary constant. Hence, we have found the solution $y(t) = y_i(t)$ of (3.1) with (2.1) and (3.2).

The solution of

$$y(t+1) = \lambda y(t) + p(t)\lambda^t\,, \tag{3.3}$$

where p is a polynomial of degree $i-1$, can be reduced to the solution of (3.1) by means of linear combinations. Successive application of the solution of (3.3) for $i = 1, 2, 3, \ldots$ again leads to the solution of the system (1.3) constructed in the foregoing preposition.

References

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