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On the König and the dual König property of order-interval hypergraphs of series-parallel posets

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ABSTRACT. Let P be a finite poset. We consider the hypergraph $\mathcal{H}(P)$ whose vertices are the elements of P and whose edges are the maximal intervals of P. We prove the König and the dual König property of $\mathcal{H}(P)$ for the class of series-parallel posets.

1 Introduction

Let P be a finite poset. A subset I of P of the form $I = \{v \in P : p \leq v \leq q\}$ (denoted $[p, q]$) is called an *interval*. If p resp. q is a minimal resp. maximal element of P, then $[p, q]$ is called *maximal interval*. Let $\mathcal{I}(P)$ be the family of maximal intervals of P. The hypergraph $\mathcal{H}(P) = (P, \mathcal{I}(P))$, briefly denoted by $\mathcal{H} = (P, \mathcal{I})$, whose vertices are the elements of P and whose edges are the maximal intervals of P is said to be the *order-interval hypergraph of P*. The *representative graph* or *line-graph* $L(\mathcal{H})$ of $\mathcal{H}$ is a graph whose vertices are points $e_1, \dots, e_m$ representing the edges $I_1, \dots, I_m$ of $\mathcal{H}(P)$, the vertices e_i, e_j being adjacent iff $I_i \cap I_j \neq \emptyset$. The dual $\mathcal{H}^*$ of the order-interval hypergraph $\mathcal{H}$ is a hypergraph whose vertices $e_1, \dots, e_m$ correspond to intervals of P and with edges $X_i = \{e_j : \quad x_i \in I_j\}$. The line-graph $L(\mathcal{H}^*)$ of the dual of the order-interval hypergraph of P is called *the representative graph* of P. More precisely, the vertices of $L(\mathcal{H}^*)$ are the points of P and two vertices are joined iff they belong to the same interval of P.

Let α, ν, ρ and τ be the independence, matching, edge-covering, and vertex-covering number of a hypergraph $\mathcal{H}$, respectively. $\mathcal{H}$ has the König property if $\nu(\mathcal{H}) = \tau(\mathcal{H})$ and it has the dual König property if $\nu(\mathcal{H}^*) = \tau(\mathcal{H}^*)$ (or $\alpha(\mathcal{H}) = \rho(\mathcal{H})$ since $\alpha(\mathcal{H}) = \nu(\mathcal{H}^*)$ and $\rho(\mathcal{H}) = \tau(\mathcal{H}^*)$). Several interesting results exist about the matching, covering, independence and chromatic numbers of $\mathcal{H}$ such as the algorithmic complexity and min-max relations (see [2, 3, 4, 5]).

Let P and Q be two posets. The *disjoint sum* $P + Q$ of P and Q is the poset on the union $P \cup Q$, such that $x \leq y$ in $P + Q$ if either $x, y \in P$ and $x \leq_P y$, or $x, y \in Q$ and $x \leq_Q y$. The *linear sum* $P \oplus Q$ of P and Q is the poset on the union $P \cup Q$, such that $x \leq y$ in $P \oplus Q$ if either $x, y \in P$ and $x \leq_P y$, or $x, y \in Q$ and $x \leq_Q y$, or $x \in P$ and $y \in Q$. In [3], it is proved that for the class of interval orders, i.e. the posets which do not contain a subposet isomorphic to the disjoint sum of two chains of length 1, $\mathcal{H}$ has the König and the dual König properties. We are interested in this paper in another class of posets called *series-parallel*, namely they can be constructed from singletons using the operations of disjoint sum and linear sum. Series-parallel posets may be characterized by the fact that they do not contain a subset isomorphic to N, [8, 9] (N represents a subset of four elements $\{x, y, z, t\}$ such that $x < y > z < t$ and these are the only comparability relations). We prove that if P is series-parallel then $\mathcal{H}(P)$ has the König and dual König properties.

Theorem 1 *The representative graph $L(\mathcal{H}^*)$ of a series-parallel poset is perfect.*

Proof: We prove the perfectness of $L(\mathcal{H}^*)$ by using Seinsche's Theorem [7] which states that every graph with no induced P_4 (elementary chain of 4 vertices without chords) is perfect. Assume that there is an induced P_4, say $\mu = [x_1, x_2, x_3, x_4]$. In P, $x_1 \parallel x_3$, $x_1 \parallel x_4$ and $x_2 \parallel x_4$.
Let $I_1 = [p_1, q_1]$ be the interval containing both x_1 and x_2, $I_2 = [p_2, q_2]$ the interval containing both x_2 and x_3, and $I_3 = [p_3, q_3]$ the interval containing both x_3 and x_4. Let us distinguish three cases.
Case 1. $x_1 < x_2$.
Since the chain μ is chordless and the vertices x_2 and x_3 belong to the same interval in P, we have that $p_2 < x_2$, $x_3 < q_2$ with $x_2 \parallel x_3$ or $x_2 > x_3$.
Case 1.1. $x_2 \parallel x_3$.
Then the poset induced by $\{x_1, x_2, p_2, x_3\}$ and N are isomorphic. It suffices to verify that $p_2 \parallel x_1$ which is true because otherwise x_1x_3 is a chord of μ.
Case 1.2. $x_2 > x_3$.
Then the poset induced by $\{x_1, x_2, x_3, x_4\}$ and N are isomorphic if $x_3 < x_4$. In the other situation, i.e. when $x_3 \parallel x_4$, we have that $\{x_1, x_2, p_3, x_4\}$ and N are isomorphic. Indeed, $p_3 < x_2$ since $p_3 \leq x_3 < x_2$ and $x_1 \parallel p_3$ since $p_3 \leq x_1 < x_2$ implies the existence of the chord x_1x_3.
Case 2. $x_2 < x_1$.
By duality, this case is similar to the first case.
Case 3. $x_1 \parallel x_2$. We have three possibilities:
Case 3.1. $x_3 > x_2$.
Then the poset induced by $\{x_1, q_1, x_2, x_3\}$ is isomorphic to N since x_1x_3 is not a chord of μ, hence $q_1 \parallel x_3$.
Case 3.2. $x_3 < x_2$.
Then the poset induced by $\{x_3, x_2, p_1, x_1\}$ is isomorphic to N since $p_1 \leq x_3$ implies $p_1 \leq x_3 < x_2 \leq q_1$, i.e. x_1x_3 is a chord of μ.

Case 3.3. $x_3 \parallel x_2$.
Because it is not possible to have simultaneously $x_1 > p_2$ and $x_1 < q_2$, we may infer that the poset induced by $\{p_2, x_2, p_1, x_1\}$ is isomorphic to N if $x_1 \not> p_2$ and the poset induced by $\{x_1, q_1, x_2, q_2\}$ is isomorphic to N if $x_1 \not< q_2$. □

Let $\mathcal{H} = (E_1, \ldots, E_m)$ be a hypergraph. We say that $\mathcal{H}$ has the *Helly property* or is a *Helly hypergraph* if every intersecting family of $\mathcal{H}$ is a star, i.e. for $J \subset \{1, \ldots, m\}$, $E_i \cap E_j \neq \emptyset$, for $i, j \in J$ implies $\bigcap_{j \in J} E_j \neq \emptyset$. A good characterization of a Helly hypergraph, due to Berge and Duchet [1], is given by the following property:

For any three vertices a_1, a_2, a_3 the family of edges containing at least two of the vertices a_i has a non-empty intersection. This characterization will be used in the following theorem:

Theorem 2 *Let $\mathcal{H} = (P, \mathcal{I})$ be the order-interval hypergraph of a series-parallel order. Then $\mathcal{H}$ is a Helly hypergraph.*

Proof: Let $\mathcal{I} = \{I_1, \ldots, I_m\}$ be the family of maximal intervals of P. We suppose that there exist three vertices a_1, a_2, a_3 of P such that $\bigcap_{j \in J} I_j = \emptyset$, where $J = \{j : \quad |I_j \cap \{a_1, a_2, a_3\}| \geq 2\}$.
Then $|J| \geq 3$ and there exist three edges, say w.l.o.g., $I_1 = [p_1, q_1]$, $I_2 = [p_2, q_2]$ and $I_3 = [p_3, q_3]$, such that

$$\begin{array}{lll} a_2, a_3 \in I_1 & \text{and} & a_1 \notin I_1 \\ a_1, a_3 \in I_2 & \text{and} & a_2 \notin I_2 \\ a_1, a_2 \in I_3 & \text{and} & a_3 \notin I_3 \end{array}$$

Since not simultaneously $p_3 \leq a_3$ and $a_3 \leq q_3$ let w.l.o.g. $p_3 \not\leq a_3$.
Case 1. $a_1 \parallel a_3$.
Then the poset induced by $\{p_3, a_1, p_2, a_3\}$ and N are isomorphic. It suffices to verify that $p_2 \neq p_3$ which is true because $a_3 \not\geq p_3$ but $a_3 > p_2$.
Case 2. $a_1 < a_3$.
Then we have $p_3 \leq a_1 < a_3$, a contradiction.
Case 3. $a_3 < a_1$.
We distinguish three subcases:
Case 3.1. $a_1 < a_2$.
Then $a_1 \in I_1$, a contradiction.
Case 3.2. $a_2 < a_1$.
Then the poset induced by $\{p_3, a_2, p_1, a_3\}$ is isomorphic to N. Indeed, we have $a_2 \parallel a_3$ since $a_2 < a_3$ gives $a_3 \in I_3$ and $a_3 < a_2$ gives $a_2 \in I_2$; $p_1 \neq p_3$ since $a_3 \not\geq p_3$ and $a_3 > p_1$.

Case 3.3. $a_1 \parallel a_2$.
Then the poset induced by $\{a_1, q_3, a_2, q_1\}$ is isomorphic to N. Indeed, we have $a_1 \not\leq q_1$ because $p_1 \leq a_3 < a_1 \leq q_1$ gives $a_1 \in I_1$; $q_1 \neq q_3$ since $a_1 < q_3$ and $a_1 \not\leq q_1$. □

Theorem 3 *The dual $\mathcal{H}^*$ of the order-interval hypergraph $\mathcal{H}$ of a series-parallel order has the Helly property.*

Proof: A necessary and sufficient condition for a hypergraph $\mathcal{H}^*$ to have the Helly property is that for any three edges I_1, I_2, I_3 of $\mathcal{H}$, there exists an edge (of $\mathcal{H}$) containing the set $(I_1 \cap I_2) \cup (I_1 \cap I_3) \cup (I_2 \cap I_3)$ [1]. For our hypergraph, we claim that $I_1 \cap I_2 \subseteq I_3$ or $I_1 \cap I_3 \subseteq I_2$ or $I_2 \cap I_3 \subseteq I_1$. Suppose not. Then we have $I_1 \cap I_2 \not\subseteq I_3$, $I_1 \cap I_3 \not\subseteq I_2$ and $I_2 \cap I_3 \not\subseteq I_1$. Let a_1, a_2 and a_3 be three vertices of P such that

$$\begin{aligned} a_1 \in I_2 \cap I_3 \quad &\text{and} \quad a_1 \notin I_1 \\ a_2 \in I_1 \cap I_3 \quad &\text{and} \quad a_2 \notin I_2 \\ a_3 \in I_1 \cap I_2 \quad &\text{and} \quad a_3 \notin I_3 \end{aligned}$$

i.e.

$$\begin{aligned} a_2, a_3 \in I_1 \quad &\text{and} \quad a_1 \notin I_1 \\ a_1, a_3 \in I_2 \quad &\text{and} \quad a_2 \notin I_2 \\ a_1, a_2 \in I_3 \quad &\text{and} \quad a_3 \notin I_3. \end{aligned}$$

But, we have already seen in the proof of Theorem 2, this leads to a contradiction. □

$\mathcal{H}$ is said to be normal if every partial hypergraph $\mathcal{H}'$ has the coloured edge property, i.e. it is possible to colour the edges of $\mathcal{H}'$ with $\Delta(\mathcal{H}')$ colours, where $\Delta(\mathcal{H}')$ represents the maximum degree of $\mathcal{H}'$. The normality of $\mathcal{H}$ induces the same property for all its partial hypergraphs. Several sufficient conditions exist for a hypergraph to have the König property [1]. One of them is its normality. A hypergraph $\mathcal{H}$ is normal iff it satisfies the Helly property and $L(\mathcal{H})$ is a perfect graph. This enables us to present the following corollary:

Corollary *If P is a series-parallel poset then every subhypergraph of $\mathcal{H}(P)$ has the dual König property.*

Proof: The line graph $L(\mathcal{H}^*)$ is a perfect graph by Theorem 1 and $\mathcal{H}^*$ has the Helly property by Theorem 3. Hence, $\mathcal{H}^*$ is normal and consequently every partial hypergraph is also normal. Since the dual of a partial hypergraph of $\mathcal{H}^*$ is a subhypergraph of $\mathcal{H}$, we may infer that every subhypergraph of $\mathcal{H}$ has the dual König property. □

Theorem 4 *If P is a series-parallel poset then $\mathcal{H}(P)$ has the König property.*

Proof: We are interested only in the case where P is connected, because if P has not this property, it can be decomposed into k connected subposets and the relations $\nu(\mathcal{H}(P)) = \nu(\mathcal{H}(P_1)) + ... + \nu(\mathcal{H}(P_k))$, $\tau(\mathcal{H}(P)) = \tau(\mathcal{H}(P_1)) + ... + \tau(\mathcal{H}(P_k))$ may be used.
Let P be a connected poset not reduced to a singleton. P can be decomposed into subposets using only the linear sum, say $P = \bigoplus_{i=1}^{k} P_i$, with $k \geq 2$.
The case $k \geq 3$ implies directly $\nu(\mathcal{H}) = \tau(\mathcal{H}) = 1$ because each singleton with one element of P_2 represents a vertex-covering of $\mathcal{H}$.
Let $k = 2$. Denote by $P_{11}, ..., P_{1k_1}$ and $P_{21}, ..., P_{2k_2}$ the connected components of P_1 and P_2, respectively. We suppose w.l.o.g. that $k_1 \leq k_2$.
Let $j_0 \in \{1, ..., k_1\}$ and a_{ij_0} (resp. b_{ij_0}) any minimal (resp. maximal) element of P_{ij_0}. Since P_{ij_0} is connected and series-parallel, each maximal element (of P_{ij_0}) is above every minimal element (of P_{ij_0}). It follows, $p < b_{1j_0}$ for all $p \in \min(P_{1j_0})$. Also, we have by the $\oplus$-operation $b_{1j_0} < p'$ for all $p' \in \min(P_2)$. Hence, each edge of $\mathcal{H}(P_{1j_0} \oplus P_2)$ contains the point b_{1j_0} and more generally, for $P = (P_{11} + ...+ P_{1k_1}) \oplus P_2$, the set $T = \{b_{1j}, j = 1, ..., k_1\}$ represents a point cover of $\mathcal{H}(P)$. On the other hand, it is not difficult to see that the family of edges $\{[a_{1j}, b_{2j}], j = 1, ..., k_1\}$ forms a matching of $\mathcal{H}$. It follows $\nu(\mathcal{H}) = \tau(\mathcal{H}) = \min\{k_1, k_2\}$. $\square$

Remark An analogous reasoning produces another proof for the dual König property. Indeed, let $P = Q_1 + ... + Q_k$ be a series-parallel poset where each component Q_i is constructed from a $\oplus$-operation, say $Q_i = Q_{i1} \oplus Q_{i2}$. Denote by $\min(Q_i) = \{a_{i1}, \ldots, a_{ik}\}$ (resp. $\max(Q_i) = \{b_{i1}, \ldots, b_{il}\}$) the set of minimal (resp. maximal) elements of Q_i and consider the family of edges $\mathcal{I}_i^{(k)}$ of $\mathcal{H}(Q_i)$ such that $\mathcal{I}_i^{(k)} = \{[a_{ij}, b_{ij}],\ \ j = 1, \ldots, k\}$. Let $\mathcal{R}_i$ be $\mathcal{I}_i^{(k)}$ if $k = l$, $\mathcal{I}_i^{(k)} \cup \{[a_{ik}, b_{ij}],\ \ j = k+1, \ldots, l\}$ if $k < l$ and $\mathcal{I}_i^{(l)} \cup \{[a_{ij}, b_{il}],\ \ j = l+1, \ldots, k\}$ if $l < k$. It is not difficult to see that $\mathcal{R}_i$ is an edge-covering of $\mathcal{H}(Q_i)$ of size $\max\{|\min(Q_i)|, |\max(Q_i)|\}$. It follows, $\alpha(\mathcal{H}(Q_i)) = \rho(\mathcal{H}(Q_i)) = \max\{|\min(Q_i)|, |\max(Q_i)|\}$. Since $\alpha(\mathcal{H}(Q_i + Q_j)) = \alpha(\mathcal{H}(Q_i)) + \alpha(\mathcal{H}(Q_j))$ and $\rho(\mathcal{H}(Q_i + Q_j)) = \rho(\mathcal{H}(Q_i)) + \rho(\mathcal{H}(Q_j))$, we deduce

$\alpha(\mathcal{H}(P)) = \sum_{i=1}^{k} \alpha(\mathcal{H}(Q_i)) = \sum_{i=1}^{k} \max\{|\min(Q_i)|, |\max(Q_i)|\}$ and
$\rho(\mathcal{H}(P)) = \sum_{i=1}^{k} \rho(\mathcal{H}(Q_i)) = \sum_{i=1}^{k} \max\{|\min(Q_i)|, |\max(Q_i)|\}$.

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