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Existence of solutions for some semilinear elliptic systems

ABSTRACT. We obtain results on the existence of solutions for the semilinear elliptic equation $(-\Delta + q)u = \lambda\rho u + f(x,u)$ in $I\!R^N$ under the hypothesis that $N \geq 3,\ q(x) \to +\infty$ as $|x| \to +\infty$ and $\rho \in L^{\frac{N}{2}}(I\!R^N)$. Similar results for semilinear systems of n equations are also established.

1 Introduction

In the present paper we will study the maximum principle and existence of solutions for the following elliptic equation:

$$(E_q) \quad L_q u := (-\Delta + q)u = \lambda\rho u + f(x,u) \text{ in } I\!R^N,$$

where $0 \leq \rho \in L^{\frac{N}{2}}(I\!R^N)$, $\lambda \in I\!R$, and $q \in L^1_{loc}(I\!R^N) \cup L^2_{loc}(I\!R^N) \cup C(I\!R^N)$ is such that $q \geq \textit{constant}\ c > 0$ and $q(x) \to +\infty$ as $|x| \to +\infty$. Such topic has been studied already in [8] for the case that $\rho \in L^\infty(I\!R^N)$ using the first eigenvalue $\lambda(q)$ of the operator $-\Delta + q$ in $L^2(I\!R^N)$. Here we obtain results on existence of solutions to (E_q) in terms of the first eigenvalue $\lambda(\rho)$ of the following problem studied in [6]:

$$(E_\rho) \quad \begin{cases} -\Delta u = \lambda\rho u \text{ in } I\!R^N \\ u(x) \to 0 \text{ as } |x| \to +\infty \end{cases}$$

We study also the following elliptic systems on $I\!R^N$:

$$L_{q_i} u_i := (-\Delta + q_i)u_i = \lambda_i \rho_i + \sum_{j=1; j\neq i}^{n} a_{ij} u_j + f_i \text{ in } I\!R^N \quad (1 \leq i \leq n).$$

Under mild assumptions we have obtained the existence of solutions to the above systems.

2 Recalls

2.1 About one equation

Let $\mathcal{D}(I\!R^N) = \mathcal{C}_0^\infty(I\!R^N) = \mathcal{C}_c^\infty(I\!R^N)$ be the set of $\mathcal{C}^\infty$ functions on $I\!R^N$ with compact support. Let $3 \leq N \in I\!N; 0 \leq \rho \in L^{\frac{N}{2}}(I\!R^N), \rho \neq 0$. Let $D^{1,2}$ be the completion of $\mathcal{D}(I\!R^N)$ under the norm $[\int_{I\!R^N} |\nabla u|^2]^{\frac{1}{2}}$.

We recall from [6] that the following equation

$$(E_\rho) \quad \begin{cases} -\Delta u = \lambda \rho u \text{ in } I\!R^N \\ u(x) \to 0 \text{ as } |x| \to +\infty \end{cases}$$

admits a simple and positive eigenvalue $\lambda(\rho)$, called the principal eigenvalue, associated with a positive eigenfunction ψ_ρ, such that

$$\lambda(\rho) \cdot \int_{I\!R^N} \rho u^2 \leq \int_{I\!R^N} |\nabla u|^2 \quad (\forall u \in D^{1,2}).$$

2.2 Schrödinger operators

Let $q \in L^1_{loc}(I\!R^N) \cup L^2_{loc}(I\!R^N) \cup C(I\!R^N)$ be such that: $q \geq constant\ c > 0$ and $q(x) \to +\infty$ as $|x| \to +\infty$, cf. [2, 3, p.3, 44, 68] and [15, Theorem XIII.47, p.207]. The variational space is the Hilbert space $V_q(I\!R^N)$ which is the completion of $\mathcal{D}(I\!R^N)$ under the norm $\|u\|_q = [\int_{I\!R^N} |\nabla u|^2 + q|u|^2]^{\frac{1}{2}}$.

Proposition 2.1 (see [1, Prop.I.1.1])
The embedding of $V_q(I\!R^N)$ into $L^2(I\!R^N)$ is compact and has dense range.

Proposition 2.2 (see e.g. [1, p.25 to 27])
$-\Delta + q$, considered as an operator in $L^2(I\!R^N)$, is positive, selfadjoint, with compact inverse. Its spectrum is discrete and consists of an infinite sequence of positive eigenvalues tending to $+\infty$. The smallest eigenvalue, called the principal eigenvalue and denoted by $\lambda(q)$, is simple and has a nonnegative eigenfunction ϕ_q. Moreover, there holds: $\forall u \in V_q(I\!R^N), \lambda(q) \int_{I\!R^N} |u|^2 \leq \int_{I\!R^N} [|\nabla u|^2 + q|u|^2]$.

2.3 M-matrix

We say that a matrix is *positive* if all of its entries are positive.

Definition 2.1 *A matrix $M = sI - B$ is called a non singular M-matrix if B is a positive matrix and $s > \rho(B) > 0$, where $\rho(B)$ denotes the spectral radius of B.*

Proposition 2.3 (see [4, Theorem 2.3, p.134])
If M is a matrix with nonpositive off-diagonal entries, then the following conditions $(P0)$, $(P1)$, $(P2)$, $(P3)$, $(P4)$ are equivalents:

$(P0)$ *M is a non singular M-matrix.*
$(P1)$ *All the principal minors of M are strictly positives.*
$(P2)$ *M is semi-positive i.e.: $\exists X >> 0$ such that $MX >> 0$.*
Here $X >> 0$ means that the entries of X are strictly positive.
$(P3)$ *M has a positive inverse.*
$(P4)$ *There exists a diagonale matrix D, $D > 0$, such that $MD + D^tM$ is positive definite.*

3 The scalar case

3.1 Case of a linear equation

We consider the following elliptic equation:

$$(E_q) \quad L_q u := (-\Delta + q)u = \lambda \rho u + f \text{ in } I\!R^N$$

where $N \geq 3,\ f \in L^2(I\!R^N)$, $0 \leq \rho \in L^{\frac{N}{2}}(I\!R^N), \rho \neq 0,\ \lambda \in I\!R$ and $q \in L^1_{loc}(I\!R^N) \cup L^2_{loc}(I\!R^N) \cup C(I\!R^N)$ is such that $q \geq$ *constant* $c > 0$ and $q(x) \to +\infty$ as $|x| \to +\infty$. We say that u is a *weak solution* of (E_q) if $u \in V_q(I\!R^N)$ and

$$\int_{I\!R^N} (\nabla u.\nabla\phi + qu\phi) = \lambda \int_{I\!R^N} \rho u\phi + \int_{I\!R^N} f\phi \quad (\forall \phi \in D(I\!R^N)).$$

Theorem 3.1 *Assume $\lambda < \lambda(\rho)$. Then (E_q) admits a unique weak solution $u \in V_q(I\!R^N)$. Moreover, if $f \geq 0$, then $u \geq 0$.*

The proof of Theorem 3.1 is elmentary and thus the details are omitted.

By using a theorem obtained in [8] we can derive the following result for the asymptotic behaviour of the solutions.

Theorem 3.2 *Let u be a weak solution of (E_q). Then there holds the convergence that $\lim_{|x|\to\infty} u(x) = 0$ under each of the following conditions* **a)-b)***:*

a) *$N = 3$.*

b) *$N \geq 3$ and either $|f| \in H^m(I\!R^N)$ with $m > \frac{N}{2} - 2$ or $f \in L^{\frac{d}{2}}(I\!R^N)$ with $d > N$.*

3.2 Case of a semilinear equation

Of concern is the following semilinear elliptic equation

$$(E_q) \quad L_q u(x) := (-\Delta + q(x))u(x) = \lambda\rho(x)u(x) + f(x, u(x)) \text{ for } x \in I\!\!R^N.$$

In [8], in case the coefficient ρ is bounded, we have established already the existence of solutions for (E_q) by the sub and supper solutions method combining with the Schauder fixed point Theorem. Here we are interested in the more general case $\rho \in L^{\frac{N}{2}}(I\!\!R^N)$ to which the above method fails and thus we have to use the approximation method due to L. Boccardo, J. Fleckinger and F. de Thlin [5].

Theorem 3.3 *Let the following conditions* $(h1) - (h4)$ *be satisfied:*
$(h1)$ $N \geq 3$; $0 \leq \rho \in L^{\frac{N}{2}}(I\!\!R^N) \cap L^\infty_{loc}(I\!\!R^N), \rho \neq 0$; $\lambda \in I\!\!R$; *and* $q \in L^1_{loc}(I\!\!R^N) \cup L^2_{loc}(I\!\!R^N) \cup C(I\!\!R^N)$ *is such that* $q \geq$ *constant* $c > 0$ *and* $q(x) \to +\infty$ *as* $|x| \to \infty$.
$(h2)$ $\exists \theta \in L^2(I\!\!R^N), \forall u \in L^2(I\!\!R^N), |f(x,u)| \leq \theta$.
$(h3)$ f *is Lipschitz respect to* u, *uniformly in* x.
$(h4)$ $\lambda < \lambda(\rho)$.
Then (E_q) *has a weak solution.*

Proof: Let $\epsilon \in]0,1[$ and $B_\epsilon = B(0, \frac{1}{\epsilon}) = \{x \in I\!\!R^N,\ |x| < \frac{1}{\epsilon}\}$. Let 1_{B_ϵ} be the indicator function of B_ϵ. Let $m \in I\!\!R^{*+}$ be such that $\lambda + m > 0$.
Let $T_\epsilon : L^2(I\!\!R^N) \to L^2(I\!\!R^N)$ be defined by $T_\epsilon(u) = v$, where u, v are determined as follows:

$$(E1) \quad (-\Delta + q + m\rho)v = (\lambda + m)\frac{\rho u}{1 + \epsilon\rho|u|}1_{B_\epsilon} + f(x,u) \text{ in } I\!\!R^N.$$

i) First we prove that T_ϵ is well defined.
By $(h2)$ and the scalar case, since $-m < 0 < \lambda(\rho)$, we deduce the existence (and uniqueness) of the weak solution $v \in V_q(I\!\!R^N)$ of $(E1)$.

ii) Construction of a sub and supper solution for $(E1)$.
We have: $(\lambda + m)\frac{1}{\epsilon}1_{B_\epsilon} + \theta \in L^2(I\!\!R^N)$ so by the scalar case, we deduce that: $\exists! \xi^0_\epsilon \in V_q(I\!\!R^N)$, $\xi^0_\epsilon \geq 0$ such that: $(-\Delta + q + m\rho)\xi^0_\epsilon = (\lambda + m)\frac{1}{\epsilon}1_{B_\epsilon} + \theta$ in $I\!\!R^N$. ξ^0_ϵ is a suppersolution of $(E1)$. By the same way, $\xi_{\epsilon,0} = -\xi^0_\epsilon$ is a subsolution of $(E1)$. Note that $T_\epsilon(\sigma_\epsilon) \subset \sigma_\epsilon$ where $\sigma_\epsilon = [\xi_{\epsilon,0}, \xi^0_\epsilon]$.

iii) We prove that T_ϵ is continuous.
Let $(u_n)_{n \in I\!\!N}$ be a sequence in $\sigma_\epsilon = [\xi_{\epsilon,0}, \xi^0_\epsilon]$ converging to u in $\|.\|_{L^2(I\!\!R^N)}$. Set $T_\epsilon(u_n) = v_n$ $(n \in I\!\!N)$ and $T_\epsilon(u) = v$. We have in a weak sense:
$(-\Delta + q + m\rho)(v_n - v) = (\lambda + m)[\frac{\rho u_n}{1+\epsilon\rho|u_n|} - \frac{\rho u}{1+\epsilon\rho|u|}]1_{B_\epsilon} + f(x,u_n) - f(x,u)$.
Multiplying by $v_n - v$, integrating over $I\!\!R^N$ and using the Cauchy-Schwartz inequality,

we get:
$\|v_n - v\|^2_{q+m\rho} \le (\lambda+m)[\int_{B_\epsilon}(\frac{\rho u_n}{1+\epsilon\rho|u_n|} - \frac{\rho u}{1+\epsilon\rho|u|})^2]^{\frac{1}{2}}\|v_n - v\|_{L^2(I\!R^N)}$
$+\|f(x,u_n) - f(x,u)\|_{L^2(I\!R^N)}\|v_n - v\|_{L^2(I\!R^N)}$.
But the function l on $I\!R$ defined by $l(x) := \frac{x}{1+|x|}$ $(\forall x \in I\!R)$ is Lipschitz and satisfies: $\forall x, y \in I\!R,\ |l(x) - l(y)| \le |x - y|$. So, by $(h3)$, we deduce that
$\|v_n - v\|_{q+m\rho} \le [(\lambda+m)[\int_{B_\epsilon}\rho^2(u_n - u)^2]^{\frac{1}{2}} + k\|u_n - u\|_{L^2(I\!R^N)}]\|v_n - v\|_{L^2(I\!R^N)}$ for some positive constant k. Since $\rho \in L^\infty_{loc}(I\!R^N)$ we find that $\|v_n - v\|_{q+m\rho} \le K(\epsilon)\|u_n - u\|_{L^2(I\!R^N)}$ for some positive constant $K(\epsilon)$. Hence $(v_n)_n$ is a sequence converging to v in $\|.\|_{L^2(I\!R^N)}$.

iv) Let $(v_n)_n$ be a sequence such that: $T_\epsilon(u_n) = v_n$.
We want to prove that there exists a subsequence of (v_n) converging in $L^2(I\!R^N)$. We have: $\forall n,\ (-\Delta + q + m\rho)v_n = (\lambda+m)\frac{\rho u_n}{1+\epsilon\rho|u_n|}1_{B_\epsilon} + f(x,u_n)$ in $I\!R^N$.
By the Cauchy-Schwartz inequality and by $(h2)$, we obtain:
$\|v_n\|^2_{q+m\rho} \le (\lambda+m)\frac{1}{\epsilon}[\int_{B_\epsilon}1]^{\frac{1}{2}}\|v_n\|_{L^2(B_\epsilon)} + \|\theta\|_{L^2(I\!R^N)}\|v_n\|_{L^2(I\!R^N)}$.
So the sequence (v_n) is bounded in $V_q(I\!R^N)$. Since the embedding of $V_q(I\!R^N)$ into $L^2(I\!R^N)$ is compact, we can find a subsequence of (v_n) which is convergent in $L^2(I\!R^N)$. Therefore $T_\epsilon(\sigma_\epsilon)$ is compact.

v) By the Schauder Fixed Point Theorem, we deduce the existence of $u_\epsilon \in \sigma_\epsilon$ such that: $T_\epsilon(u_\epsilon) = u_\epsilon$. Moreover, there holds

$$(-\Delta + q + m\rho)u_\epsilon = (\lambda+m)\frac{\rho u_\epsilon}{1+\epsilon\rho|u_\epsilon|}1_{B_\epsilon} + f(x,u_\epsilon) \text{ in } I\!R^N.$$

vi) Now we prove that $(\epsilon u_\epsilon)_\epsilon$ is a bounded sequence in $V_q(I\!R^N)$.
Multiplying by $\epsilon^2 u_\epsilon$, integrating over $I\!R^N$, using the Cauchy-Schwartz inequality and since $\frac{1_{B_\epsilon}}{1+\epsilon\rho|u_\epsilon|} \le 1$ we deduce that

$$\|\epsilon u_\epsilon\|^2_{q+m\rho} \le (\lambda+m)\int_{I\!R^N}\rho\epsilon^2 u_\epsilon^2 + \|\theta\|_{L^2(I\!R^N)}\|\epsilon u_\epsilon\|_{L^2(I\!R^N)}.$$

Therefore, $\|\epsilon u_\epsilon\|^2_{q+m\rho} \le \frac{\lambda+m}{\lambda(\rho)+m}\|\epsilon u_\epsilon\|^2_{q+m\rho} + \|\theta\|_{L^2(I\!R^N)}\|\epsilon u_\epsilon\|_{L^2(I\!R^N)}$.
It follows that $0 \le \frac{\lambda(\rho)-\lambda}{\lambda(\rho)+m}\|\epsilon u_\epsilon\|^2_{q+m\rho} \le \|\theta\|_{L^2(I\!R^N)}\|\epsilon u_\epsilon\|_{L^2(I\!R^N)}$. Hence: $\exists K > 0$, K independent of ϵ such that $\|\epsilon u_\epsilon\|_{q+m\rho} \le K$.

vii) We prove now that $\epsilon u_\epsilon \to 0$ as $\epsilon \to 0$ strongly in $L^2(I\!R^N)$ and weakly in $V_q(I\!R^N)$. We know that the imbedding of $V_q(I\!R^N)$ into $L^2(I\!R^N)$ is compact. The sequence $(\epsilon u_\epsilon)_\epsilon$ is bounded in $V_q(I\!R^N)$ so (for a subsequence), we deduce that $\exists u^*$ such that: $\epsilon u_\epsilon \to u^*$ as $\epsilon \to 0$ strongly in $L^2(I\!R^N)$ and weakly in $V_q(I\!R^N)$. Multiplying by ϵ, we get:
$(-\Delta + q + m\rho)\epsilon u_\epsilon = (\lambda+m)\frac{\rho\epsilon u_\epsilon}{1+\epsilon\rho|u_\epsilon|}1_{B_\epsilon} + \epsilon f(x,u_\epsilon)$ in $I\!R^N$.

We have: $\forall\phi\in\mathcal{D}(I\!R^N),\ |\int_{I\!R^N}\epsilon f(x,u_\epsilon)\phi|\leq\epsilon\int_{I\!R^N}\theta|\phi|$.
We deduce that: $\int_{I\!R^N}\epsilon f(x,u_\epsilon)\phi\to 0$ as $\epsilon\to 0$.
We have also: $[\frac{\epsilon\rho u_\epsilon}{1+\epsilon\rho|u_\epsilon|}1_{B_\epsilon}-\frac{\rho u^*}{1+\rho|u^*|}]\phi\to 0$ as $\epsilon\to 0$.
Since $\frac{\epsilon\rho|u_\epsilon|}{1+\epsilon\rho|u_\epsilon|}\leq 1$ and $\frac{\rho|u^*|}{1+\rho|u^*|}\leq 1$ we get

$$|[\frac{\epsilon\rho u_\epsilon}{1+\epsilon\rho|u_\epsilon|}1_{B_\epsilon}-\frac{\rho u^*}{1+\rho|u^*|}]\phi|\leq 2|\phi|\in L^1(I\!R^N).$$

Using the Dominated Convergence Theorem, we deduce that

$$\int_{I\!R^N}\frac{\epsilon\rho u_\epsilon}{1+\epsilon\rho|u_\epsilon|}1_{B_\epsilon}\phi\to\int_{I\!R^N}\frac{\rho u^*}{1+\rho|u^*|}\phi \text{ as } \epsilon\to 0.$$

So u^* is a weak solution of the equation $(-\Delta+q+m\rho)u^*=(\lambda+m)\frac{\rho u^*}{1+\rho|u^*|}$ in $I\!R^N$.
We prove $u^*=0$. Multiplying by u^* and integrating over $I\!R^N$ yields

$$\int_{I\!R^N}[|\nabla u^*|^2+(q+m\rho)|u^*|^2]=(\lambda+m)\int_{I\!R^N}\frac{\rho|u^*|^2}{1+\rho|u^*|}\leq(\lambda+m)\int_{I\!R^N}\rho u^{*2}.$$

Since $\lambda(\rho)\int_{I\!R^N}\rho u^{*2}\leq\int_{I\!R^N}|\nabla u^*|^2$ we have $(\lambda(\rho)+m)\int_{I\!R^N}\rho u^{*2}\leq(\lambda+m)\int_{I\!R^N}\rho u^{*2}$.
But $\lambda(\rho)-\lambda>0$, we find that $\int_{I\!R^N}\rho u^{*2}=0$ and thus $u^*=0$ *a.e.*

viii) We prove now by contradiction that $(u_\epsilon)_\epsilon$ is bounded in $V_q(I\!R^N)$.
We suppose that (for a subsequence): $\|u_\epsilon\|_{q+m\rho}\to+\infty$ as $\epsilon\to 0$. Let $z_\epsilon=\frac{1}{\|u_\epsilon\|_{q+m\rho}}u_\epsilon$. Then $\|z_\epsilon\|_{q+m\rho}=1$ and thus $(z_\epsilon)_\epsilon$ is a bounded sequence in $V_q(I\!R^N)$. There exists z such that $z_\epsilon\to z$ as $\epsilon\to 0$ strongly in $L^2(I\!R^N)$ and weakly in $V_q(I\!R^N)$. Furthermore: $\exists h\in L^2(I\!R^N),\forall\epsilon,\ \ |z_\epsilon|\leq h$ *a.e* (for a subsequence) (see [7, p.58]) In a weak sense, we have:
$(-\Delta+q+m\rho)z_\epsilon=\frac{(\lambda+m)}{\|u_\epsilon\|_{q+m\rho}}\frac{\rho u\epsilon}{1+\epsilon\rho|u_\epsilon|}1_{B_\epsilon}+\frac{1}{\|u_\epsilon\|_{q+m\rho}}f(x,u_\epsilon)$ in $I\!R^N$.
We have: $\forall\phi\in\mathcal{D}(I\!R^N),\ \int_{I\!R^N}\frac{1}{\|u_\epsilon\|_{q+m\rho}}f(x,u_\epsilon)\phi\to 0$ as $\epsilon\to 0$.
We have also: $[\frac{\rho z_\epsilon}{1+\epsilon\rho|u_\epsilon|}1_{B_\epsilon}-\rho z]\phi\to 0$ as $\epsilon\to 0$ *a.e.*
$|[\frac{\rho z_\epsilon}{1+\epsilon\rho|u_\epsilon|}1_{B_\epsilon}-\rho z]\phi|\leq|\rho(z_\epsilon 1_{B_\epsilon}-z)||\phi|+|\frac{\epsilon\rho|u_\epsilon|}{1+\epsilon\rho|u_\epsilon|}\rho z\phi|\leq|\rho(|z|+|h|)\phi|+|\rho z\phi|\in L^1(I\!R^N)$ since $\rho\in L^\infty_{loc}(I\!R^N)$. Using the Dominated Convergence Theorem, we deduce that $\int_{I\!R^N}\frac{\rho z_\epsilon}{1+\epsilon\rho|u_\epsilon|}1_{B_\epsilon}\phi\to\int_{I\!R^N}\rho z\phi$ as $\epsilon\to 0$. So z is a weak solution of the equation $(-\Delta+q)z=\lambda\rho z$ in $I\!R^N$. Since $\lambda<\lambda(\rho)$ we find $z=0$. Using $\|z_\epsilon\|_{L^2(I\!R^N)}=1\to\|z\|_{L^2(I\!R^N)}=0$ as $\epsilon\to 0$, we get a contradiction.

ix) There exists u^0 such that $u_\epsilon\to u^0$ strongly in $L^2(I\!R^N)$ and weakly in $V_q(I\!R^N)$.
We have in a weak sense:
$(-\Delta+q+m\rho)u_\epsilon=(\lambda+m)\frac{\rho u_\epsilon}{1+\epsilon\rho|u_\epsilon|}1_{B_\epsilon}+f(x,u_\epsilon)$ in $I\!R^N$.
But $u_\epsilon\rightharpoonup u^0$ as $\epsilon\to 0$ weakly in $V_q(I\!R^N)$. Furthermore: $\exists h'\in L^2(I\!R^N),\ \forall\epsilon,\ |u_\epsilon|\leq h'$ *a.e* (for a subsequence.) By $(h3)$ we have also:

$$\|f(x,u_\epsilon)-f(x,u^0)\|^2_{L^2(I\!R^N)}\leq const\cdot\|u_\epsilon-u^0\|^2_{L^2(I\!R^N)}.$$

Therefore: $\forall \phi \in \mathcal{D}(I\!R^N)$, $\int_{I\!R^N} f(x,u_\epsilon)\phi \to \int_{I\!R^N} f(x,u^0)\phi$ as $\epsilon \to 0$.
Moreover: $[\frac{\rho u_\epsilon}{1+\epsilon\rho|u_\epsilon|}1_{B_\epsilon} - \rho u^0]\phi \to 0$ as $\epsilon \to 0$ *a.e.* and
$|[\frac{\rho u_\epsilon}{1+\epsilon\rho|u_\epsilon|}1_{B_\epsilon} - \rho u^0]\phi| \leq \rho(|h'| + 2|u^0|)|\phi| \in L^1(I\!R^N)$.
Using the Dominated Convergence Theorem, we deduce that $\int_{I\!R^N} \frac{\rho u_\epsilon}{1+\epsilon\rho|u_\epsilon|}1_{B_\epsilon}\phi \to \int_{I\!R^N} \rho u^0\phi$ as $\epsilon \to 0$. So u^0 is a weak solution of the equation $(-\Delta + q)u^0 = \lambda\rho u^0 + f(x,u^0)$ in $I\!R^N$.

4 Study of a system

The system which we will study has the form

$$L_{q_i}u_i := (-\Delta + q_i)u_i = \lambda_i\rho_i + \sum_{j=1;j\neq i}^{n} a_{ij}u_j + f_i \text{ in } I\!R^N \quad (1 \leq i \leq n). \tag{1}$$

We assume $N \geq 3$ and impose the following conditions (H0)-(H3) for each index i :

(H0) $\lambda_i \in I\!R$.

(H1) $0 \leq \rho_i \in L^{N/2}(I\!R^N)$, $\rho_i \not\equiv 0$.

(H2) $q_i \in L^1_{loc}(I\!R^N) \cup L^2_{loc}(I\!R^N) \cup C(I\!R)$ is such that $q_i \geq c_i$ for some positive constant c_i and $q_i(x) \to +\infty$ as $|x| \to \infty$.

(H3) $f_i \in L^2(I\!R^N)$.

We say that $(u_1, ..., u_n) \in V_{q_1}(I\!R^N) \times ... \times V_{q_n}(I\!R^N)$ is a weak solution of System (1) if: $\forall 1 \leq i \leq n, \forall \phi \in \mathcal{D}(I\!R^N)$,
$\int_{I\!R^N}[\nabla u_i.\nabla\phi + q_iu_i\phi] = \lambda_i \int_{I\!R^N} \rho_iu_i\phi + \int_{I\!R^N} \sum_{j=1;j\neq i}^n a_{ij}u_j\phi + \int_{I\!R^N} f_i\phi.$

We call System (1) *cooperative* if $a_{ij} \geq 0$ for all $i \neq j$. We say that System (1) satisfies the maximum principle if: $\forall f_i \geq 0, 1 \leq i \leq n$, each solution $u = (u_1, ..., u_n)$ of (1) is nonnegative.

4.1 Study of a linear system

Theorem 4.1 *Assume conditions* $(H0) - (H3)$ *and the following ones*
$(H4)$ $\forall i, j$ $i \neq j \Rightarrow 0 \leq a_{ij} \leq k_{ij}\sqrt{\rho_i}\sqrt{\rho_j}$ *with* $k_{ij} \in I\!R^+$.
Let $D = (d_{ij})$ *be the* $n \times n$ *matrix given by* $d_{ii} = \lambda(\rho_i) - \lambda_i$ *and* $d_{ij} = -k_{ij}$ *otherwise. If* D *is a non singular M-matrix, then System* (1) *satisfies the maximum principle.*

Proof: Assume that: $\forall 1 \leq i \leq n, f_i \geq 0$. Let $(u_1, ..., u_n)$ be a weak solution of (1) and $u_i^- = \max(0, -u_i)$. Using $(H4)$ and the Cauchy-Schwartz inequality, we obtain

$$(\lambda(\rho_i) - \lambda_i)(\int_{I\!R^N} \rho_i |u_i^-|^2)^{\frac{1}{2}} - \sum_{j;j\neq i} k_{ij} (\int_{I\!R^N} \rho_j |u_j^-|^2)^{\frac{1}{2}} \leq 0.$$

Let ${}^tX = (x_1, ..., x_n)$ be such that: $\forall 1 \leq i \leq n,\ x_i = (\int_{I\!R^N} \rho_i |u_i^-|^2)^{\frac{1}{2}}$. We have $DX \leq 0$. Since D is a non singular M-matrix, by $(P3)$, D has a positive inverse. Hence, $X \leq 0$. It follows that $u_i^- \equiv 0$ for all i.

Theorem 4.2 *Assume conditions $(H0) - (H3)$ and the following ones*
$(H4*)$ $\forall i, j\ i \neq j \Rightarrow |a_{ij}| \leq k_{ij}\sqrt{\rho_i}\sqrt{\rho_j}$ *with* $k_{ij} \in I\!R^+$.
If D (given in Theorem 4.1) is a non singular M-matrix, then System (1) has a unique weak solution $(u_1, ..., u_n)$. Furthermore if System (1) is cooperative and if $f_i \geq 0$ for all i, then, by the maximum principle, we have that $u_i \geq 0$ for all i.

Proof: By $(P4)$, there exists a diagonale matrix E such that ${}^tDE + ED$ is positive definite, with: $E = (e_{ij})$ where $\forall 1 \leq i \leq n,\ e_{ii} = e_i > 0$. Let $\forall i,\ m_i \in I\!R^{*+}$ such that: $\forall 1 \leq i \leq n, \lambda_i + m_i > 0$. Let $l : (V_{q_1}(I\!R^N) \times ... \times V_{q_n}(I\!R^N))^2 \to I\!R$ be defined by
$l((u_1, ..., u_n), (v_1, ..., v_n)) = \sum_{i=1}^n e_i \int_{I\!R^N} [\nabla u_i . \nabla v_i + (q_i + m_i \rho_i) u_i v_i]$
$- \sum_{i=1}^n e_i (\lambda_i + m_i) \int_{I\!R^N} \rho_i u_i v_i - \sum_{i,j;i\neq j} e_i \int_{I\!R^N} a_{ij} u_j v_i.$
l is a bilinear continuous form. By $(H4*)$ and the Cauchy-Schwartz inequality, we prove that l is coercive. So, using the Lax-Milgram Theorem, we obtain a unique solution for System (1).

4.2 Study of a semilinear system

Theorem 4.3 *Keep conditions $(H0) - (H3)$ and $(H4*)$ and assume that the condition $(H1)$ is strengthened as follows: $\rho_i \in L^\infty_{loc}(I\!R^N) \cap L^1(I\!R^N)$ for each i. Assume further the following conditions $(H6) - (H7)$:*
$(H6)$ $\forall i,\ \exists \theta_i \in L^2(I\!R^N),\ \forall u_1, ..., u_n \in L^2(I\!R^N),\ |f_i(x, u_1, ..., u_n)| \leq \theta_i$.
$(H7)$ $\forall i,\ f_i$ *is Lipschitz respect to u_i, uniformly in x.*
If D (given in Theorem 4.1) is a non singular M-matrix, then System (1) has a weak solution $(u_1, ..., u_n)$.

Since the proof of Theorem 4.3 is very similar to that of Theorem 3.3, we omit the details.

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