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Existence of solutions for non necessarily cooperative systems involving Schrödinger operators

ABSTRACT. We study the existence of a solution for a non necessarily cooperative system of n equations involving Schrödinger operators defined on $\mathbb{R}^N$ and we study also a limit case (the Fredholm Alternative). We derive results for semilinear systems.

KEY WORDS AND PHRASES. Schrödinger operators, M-matrices, non necessarily cooperative systems.

1 Introduction

We consider the following elliptic system defined on $\mathbb{R}^N$:

$$(1)\begin{cases} \text{for } 1 \le i \le n, \\ (1i) \quad L_{q_i}u_i := (-\Delta + q_i)u_i = \sum_{j=1}^n a_{ij}u_j + f_i \text{ in } \mathbb{R}^N \end{cases}$$

where :

. n and N are two integers not equal to 0

. Δ is the Laplacian operator

(H1) for $1 \le i, j \le n,\ a_{ij} \in L^\infty(\mathbb{R}^N)$

(H2) for $1 \le i \le n$, q_i is a continuous potential defined on $\mathbb{R}^N$ such that:
$q_i(x) \ge 1, \quad \forall x \in \mathbb{R}^N$ and $q_i(x) \to +\infty$ when $|x| \to +\infty$

(H3) for $1 \le i \le n$, $f_i \in L^2(\mathbb{R}^N)$

We do not make here any assumptions on the sign of a_{ij}.
Recall that System (1) is called cooperative if $a_{ij} \ge 0$ pp for $i \ne j$.

Our paper is organized as follow:
- in Section 2, we recall some results about M-matrices and about the Maximum Principle for cooperative systems involving Schrödinger operators $-\Delta + q_i$ in $I\!R^N$.
- in Section 3, we show the existence of a solution for a non necessarily cooperative system of n equations. After that we study a limit case (the Fredholm Alternative) and finally we study the existence of a solution for a (non necessarily cooperative) semilinear system.

2 Recalls

2.1 M-matrix

We recall some results about the M-matrix([8]th2.3 p.134).
We say that a matrix is positive if all its coefficients are nonnegative and we say that a symmetric matrix is positive definite if all its principal minors are stictly positive.

Definition 2.1 ([8]) *A matrix $M = sI - B$ is called a non singular M-matrix if B is a positive matrix (i.e. with nonnegative coefficients) and $s > \rho(B) > 0$ the spectral radius of B.*

Proposition 2.2 ([8]) *If M is a matrix with nonpositive off-diagonal coefficients, the conditions $(P0), (P1), (P2), (P3), (P4)$ are equivalent.*

$(P0)$ *M is a non singular M-matrix,*

$(P1)$ *all the principal minors of M are strictly positive,*

$(P2)$ *M is semi-positive i.e.: $\exists X >> 0$ such that $MX >> 0$.*
$X >> 0$ signify $\forall i, X_i > 0$ if $X = (X_1, ..., X_n)$.

$(P3)$ *M has a positive inverse.*

$(P4)$ *there exists a diagonal matrix D, $D > 0$, such that $MD + D^tM$ is positive definite.*

Remark: If M is a non singular M-matrix, then ${}^tM$ is also a non singular M-matrix.
So $(P4) \Leftrightarrow (P5)$ with $(P5)$: there exists a diagonal matrix D, $D > 0$ such that ${}^tMD + DM$ is positive definite.

2.2 Schrödinger operators

Let $\mathcal{D}(I\!R^N) = \mathcal{C}_0^\infty(I\!R^N) = \mathcal{C}_c^\infty(I\!R^N)$ the set of functions $\mathcal{C}^\infty$ on $I\!R^N$ with compact support.
Let q a continuous potential defined on $I\!R^N$ such that :

$q(x) \geq 1, \quad \forall x \in I\!R^N$ and $q(x) \to +\infty$ when $|x| \to +\infty$. The variational space is $V_q(I\!R^N)$: the completion of $\mathcal{D}(I\!R^N)$ for the norm $\|.\|_q$ where $\|u\|_q = [\int_{I\!R^N} |\nabla u|^2 + q|u|^2]^{\frac{1}{2}}$.

$$V_q(I\!R^N) = \{u \in H^1(I\!R^N), \sqrt{q}u \in L^2(I\!R^N)\}$$

$(V_q(I\!R^N), \|.\|_q)$ is an Hilbert space.([1] prop.I.1.1)
Moreover:

Proposition 2.3 ([1] propI.1.1; [21]prop1, p.356)
The embedding of $V_q(I\!R^N)$ into $L^2(I\!R^N)$ is compact and with dense range.

To the form :

$$a(u,v) = \int_{I\!R^N} \nabla u.\nabla v + \int_{I\!R^N} quv, \qquad \forall (u,v) \in (V_q(I\!R^N))^2,$$

we associate the operator $L_q := -\Delta + q$ defined on $L^2(I\!R^N)$ by variational methods.
Here $D(L_q)$ denotes the domain of the operator L_q.
$D(L_q) = \{u \in V_q(I\!R^N), (-\Delta + q)u \in L^2(I\!R^N)\}$ ([5]th1.1, p4)
We have : $\forall u \in D(L_q), \forall v \in V_q(I\!R^N), a(u,v) = \int_{I\!R^N} L_q u.v$.
The embedding of $D(L_q)$ into $V_q(I\!R^N)$ is continuous and with dense range.
([1], p.24;[5], p.5,6)

Proposition 2.4 ([1], p.25 to 27;[5]th1.1, p.4,p.6,8,11; [6], p.3, th3.2 p.45; [14], p.488,489;[26], p.346 to 350; [28]thXIII.16, p.120,thXIII.47 p.207)
L_q, considered as an operator in $L^2(I\!R^N)$, is positive, selfadjoint, with compact inverse. Its spectrum is discrete and consists in an infinite sequence of positive eigenvalues tending to $+\infty$. The smallest one, denoted by $\lambda(q)$, is simple and associated with an eigenfunction ϕ_q which does not change sign in $I\!R^N$. We say that $\lambda(q)$ is a principal eigenvalue; it is positive and simple.
Furthermore, we have:

$$(R1)\begin{cases} L_q\phi_q = \lambda(q)\phi_q \ \textit{in } I\!R^N \\ \phi_q(x) \to 0 \ \textit{when } x \to +\infty; \phi_q > 0 \ \textit{in } I\!R^N; \lambda(q) > 0 \end{cases}$$

$$(R2)\ \forall u \in V_q(I\!R^N), \lambda(q)\int_{I\!R^N} |u|^2 \leq \int_{I\!R^N} [|\nabla u|^2 + q|u|^2].$$

Moreover the equality holds iff u is colinear to ϕ_q.

If $a \in L^\infty(I\!R^N)$ let: $a^* = \sup_{x \in I\!R^N} a(x), \quad a_* = \inf_{x \in I\!R^N} a(x)$ and

$$\lambda(q-a) = \inf\{\frac{\int_{I\!R^N} [|\nabla\phi|^2 + (q-a)\phi^2]}{\int_{I\!R^N} \phi^2}; \phi \in \mathcal{D}(I\!R^N); \phi \not\equiv 0\}.$$

The operator $-\Delta + q - a$ in $I\!R^N$ has a unique selfadjoint realization ([6] p.3) in $L^2(I\!R^N)$ which is denoted L_{q-a}. (Indeed, q is a continuous potential, $a \in L^\infty(I\!R^N)$, so the condition in [6] $(q-a)_- \in L^p_{loc}(I\!R^N)$ for a $p > \frac{N}{2}$is satisfied.)
We note also that: $\lambda(q-a) \leq \lambda(q) - a_*$ and:
$\forall m \in I\!R^{*+},\ \lambda(q-a+m) = \lambda(q-a) + m.$

The following theorem is classical:

Theorem 2.1 ([1],[11],[28]p.204) *We consider the equation:*
$(E)\ (-\Delta + q)u = au + f\, in\ I\!R^N$ *where* $a \in I\!R,\ f \in L^2(I\!R^N),\ f \geq 0$
and q *is a continuous potential on* $I\!R^N$ *such that:*
$q \geq 1$ *and* $q(x) \to +\infty$ *when* $|x| \to +\infty$.
If $a < \lambda(q)$ *then* $\exists! u \in V_q(I\!R^N)$ *solution of* (E). *Moreover, we have:* $u \geq 0$.

2.3 Cooperative systems

We consider in this section System (1) and we assume here that it is cooperative, i.e.:

(H1*) $a_{ij} \in L^\infty(I\!R^N);\ a_{ij} \geq 0\ pp$ for $i \neq j$.
We recall here a sufficient condition for Maximum Principle and existence of solutions for such cooperative systems.
We say that System (1) satisfies the Maximum Principle if: $\forall f_i \geq 0$, $1 \leq i \leq n$, any solution $u = (u_1, ..., u_n)$ of (1) is nonnegative.
Let $E = (e_{ij})$ be the matrix $n \times n$ defined by:
$\forall 1 \leq i \leq n,\ e_{ii} = \lambda(q_i - a_{ii})$ and $\forall 1 \leq i, j \leq n,\ i \neq j \Rightarrow e_{ij} = -a^*_{ij}$.

Theorem 2.2 ([11]) *We assume that* $(H1*), (H2), (H3)$ *are satisfied.*
If E *is a non singular M-matrix, then System* (1) *satisfies the Maximum Principle.*

Theorem 2.3 ([11]) *We assume that* $(H1*), (H2), (H3)$ *are satisfied.*
If E *is a non singular M-matrix and if* $f_i \geq 0$ *for each* $1 \leq i \leq n$, *then System* (1) *has a unique solution which is nonnegative.*

3 Study of a non necessarily cooperative system

3.1 Study of a non necessarily cooperative system of n equations with bounded coefficients

We adapt here an approximation method used in [9] for problems defined on bounded domains.

We consider the following elliptic system defined on $I\!R^N$:

$$(1)\begin{cases} \text{for } 1 \le i \le n, \\ (1i) \quad L_{q_i}u_i := (-\Delta + q_i)u_i = \sum_{j=1}^n a_{ij}u_j + f_i \text{ in } I\!R^N. \end{cases}$$

Let $G = (g_{ij})$ the matrix $n \times n$ defined by: $\forall 1 \le i \le n,\ g_{ii} = \lambda(q_i - a_{ii})$ and

$$\forall 1 \le i, j \le n,\ i \ne j \Rightarrow g_{ij} = -|a_{ij}|^* \text{ where } |a_{ij}|^* = \sup_{x \in I\!R^N} |a_{ij}(x)|.$$

We make the following hypothesis: (H) G is a non singular M-matrix.

Theorem 3.1 *We assume that $(H1), (H2), (H3)$ and (H) are satisfied. Then System (1) has a weak solution $(u_1, ..., u_n) \in V_{q_1}(I\!R^N) \times ... \times V_{q_n}(I\!R^N)$.*

First, we prove the following lemma:

Lemma 3.1 *We assume that $(H), (H1), (H2), (H3)$ are satisfied.*
Let $(u_1, ..., u_n) \in V_{q_1}(I\!R^N) \times ... \times V_{q_n}(I\!R^N)$ solution of:

$$(2)\begin{cases} \textit{for } 1 \le i \le n, \\ (2i) \quad L_{q_i}u_i := (-\Delta + q_i)u_i = \sum_{j=1}^n a_{ij}u_j \ \textit{in } I\!R^N. \end{cases}$$

Then: $(u_1, ..., u_n) = (0, ..., 0)$.

Proof of the lemma 3.1: Let $m \in I\!R^{*+}$ be such that:
$\forall 1 \le i \le n, m - a_{ii} > 0$. Let $q_i' = q_i + m - a_{ii} \ge 1$.
For any $1 \le i \le n$, we have:

$$\int_{I\!R^N} [|\nabla u_i|^2 + q_i'|u_i|^2] = \int_{I\!R^N} m|u_i|^2 + \sum_{j;j \ne i} \int_{I\!R^N} a_{ij}u_ju_i$$

$$\le \int_{I\!R^N} m|u_i|^2 + \sum_{j;j \ne i} \int_{I\!R^N} |a_{ij}u_ju_i|.$$

and by the characterization $(R2)$ of the first eigenvalue $\lambda(q_i')$ we get:
$(\lambda(q_i') - m) \int_{I\!R^N} |u_i|^2 \le \sum_{j;j \ne i} |a_{ij}|^* (\int_{I\!R^N} |u_j|^2)^{\frac{1}{2}} (\int_{I\!R^N} |u_i|^2)^{\frac{1}{2}}$.
So: $(\lambda(q_i') - m)(\int_{I\!R^N} |u_i|^2)^{\frac{1}{2}} \le \sum_{j;j \ne i} |a_{ij}|^* (\int_{I\!R^N} |u_j|^2)^{\frac{1}{2}}$.
Let

$$X = \begin{pmatrix} (\int_{I\!R^N} u_1^2)^{\frac{1}{2}} \\ . \\ . \\ . \\ (\int_{I\!R^N} u_n^2)^{\frac{1}{2}} \end{pmatrix}.$$

We have $X \geq 0$ and $GX \leq 0$. Since G is a non singular M-matrix, by Proposition 2.2 we deduce that $X \leq 0$. So $X = 0$ i.e. $\forall 1 \leq i \leq n, \ \ u_i = 0$.

Proof of Theorem 3.1: Let $m \in I\!R^{*+}$ such that : $\forall 1 \leq i \leq n, m - a_{ii} > 0$. Let $q_i' = q_i - a_{ii} + m \geq 1$. ($m$ exists because $\forall 1 \leq i \leq n, \ a_{ii} \in L^\infty(I\!R^N)$.)

First we note that:
$(u_1, ..., u_n) \in V_{q_1}(I\!R^N) \times ... \times V_{q_n}(I\!R^N)$ is a weak solution of (1) if and only if $(u_1, ..., u_n)$ is a weak solution of $(1')$ where:

$$(1') \begin{cases} \text{for } 1 \leq i \leq n, \\ (1'i) \quad (-\Delta + q_i')u_i = mu_i + \sum_{j;j\neq i} a_{ij}u_j + f_i \text{ in } I\!R^N . \end{cases}$$

Let $\epsilon \in]0, 1[, \ B_\epsilon = B(0, \frac{1}{\epsilon}) = \{x \in I\!R^N, \ |x| < \frac{1}{\epsilon}\}$ and 1_{B_ϵ} be the indicator function of B_ϵ.
Let $T : L^2(I\!R^N) \times ... \times L^2(I\!R^N) \to L^2(I\!R^N) \times ... \times L^2(I\!R^N)$ be defined by: $T(\xi_1, ..., \xi_n) = (\omega_1, ..., \omega_n)$ where for any $1 \leq i \leq n$,

$$(-\Delta + q_i')\omega_i = m\frac{\xi_i}{1 + \epsilon|\xi_i|}1_{B_\epsilon} + \sum_{j;j\neq i} a_{ij}\frac{\xi_j}{1 + \epsilon|\xi_j|}1_{B_\epsilon} + f_i \text{ in } I\!R^N.$$

i) First we prove that T is well defined:
Let: $\forall (\xi_1, ..., \xi_n) \in L^2(I\!R^N) \times ... \times L^2(I\!R^N), \ \ \forall 1 \leq i \leq n,$

$$\psi_i(\xi_1, ..., \xi_n) = m\frac{\xi_i}{1 + \epsilon|\xi_i|}1_{B_\epsilon} + \sum_{j;j\neq i} a_{ij}\frac{\xi_j}{1 + \epsilon|\xi_j|}1_{B_\epsilon}.$$

We have:

$$|\frac{\xi_i}{1 + \epsilon|\xi_i|}1_{B_\epsilon}| = \frac{1}{\epsilon}|\frac{\epsilon\xi_i}{1 + \epsilon|\xi_i|}1_{B_\epsilon}| \leq \frac{1}{\epsilon}1_{B_\epsilon}.$$

Since $1_{B_\epsilon} \in L^2(I\!R^N)$ and $a_{ij} \in L^\infty(I\!R^N)$, we deduce that for any $1 \leq i \leq n \ \ \psi_i(\xi_1, ..., \xi_n) \in L^2(I\!R^N)$. By $(H3)$, $f_i \in L^2(I\!R^N)$ and therefore $\psi_i(\xi_1, ..., \xi_n) + f_i \in L^2(I\!R^N)$.
By Theorem 2.1, we deduce the existence (and uniqueness) of $(\omega_1, ..., \omega_n) \in V_{q_1}(I\!R^N) \times ...V_{q_n}(I\!R^N)$. So T is well defined.

ii) We note that: $\forall (\xi_1, ..., \xi_n), \ |\psi_i(\xi_1, ..., \xi_n)| \leq n \max_{j;j\neq i}(m, |a_{ij}|^*)\frac{1}{\epsilon}1_{B_\epsilon}$.
Let $h = \frac{n}{\epsilon}\max_{i,j;i\neq j}(m, |a_{ij}|^*)1_{B_\epsilon} \in L^2(I\!R^N)$.
$h + f_i \in L^2(I\!R^N)$, so, by the scalar case, we deduce that:
$\exists! \xi_i^0 \in V_{q_i}(I\!R^N)$ such that: $(-\Delta + q_i')\xi_i^0 = h + f_i$ in $I\!R^N$.
$(\xi_1^0, ..., \xi_n^0)$ is an uppersolution of $(1')$: $\forall 1 \leq i \leq n$,

$$(-\Delta + q_i')\xi_i^0 \geq \psi_i(\xi_1, ..., \xi_n) + f_i$$

By the same way, we construct a lowersolution of $(1')$: $\forall 1 \leq i \leq n$,
$\exists ! \xi_{i,0} \in V_{q_i}(I\!\!R^N)$ such that: $(-\Delta + q_i')\xi_{i,0} = -h + f_i$ in $I\!\!R^N$.
$(\xi_{1,0}, ..., \xi_{n,0})$ is a lowersolution of $(1')$: $\forall 1 \leq i \leq n$,

$$(-\Delta + q_i')\xi_{i,0} \leq \psi_i(\xi_1, ..., \xi_n) + f_i.$$

We note that: $\forall i,\ \xi_{i,0} \leq \xi_i^0$ (because $(-\Delta + q_i')(\xi_i^0 - \xi_{i,0}) = 2h \geq 0$.)
We consider now the restriction of T, denoted by T^*, at $[\xi_{1,0}, \xi_1^0] \times ... \times [\xi_{n,0}, \xi_n^0]$. We prove now that T^* has a fixed point by the Schauder Fixed Point Theorem.

iii) First we prove that $[\xi_{1,0}, \xi_1^0] \times ... \times [\xi_{n,0}, \xi_n^0]$ is invariant by T^*.
Let $(\xi_1, ..., \xi_n) \in [\xi_{1,0}, \xi_1^0] \times ... \times [\xi_{n,0}, \xi_n^0]$. We put: $T^*(\xi_1, ..., \xi_n) = (\omega_1, ..., \omega_n)$.
We have: $(-\Delta + q_i')(\xi_i^0 - \omega_i) = h - \psi_i(\xi_1, ..., \xi_n) \geq O$.
By the scalar case, we deduce that: $\xi_i^0 \geq \omega_i$ *pp*.
By the same way we get: $(-\Delta + q_i')(\omega_i - \xi_{i,0}) = \psi_i(\xi_1, ..., \xi_n) + h \geq 0$ and $\omega_i \geq \xi_{i,0}$ *pp*.
So $[\xi_{1,0}, \xi_1^0] \times ... \times [\xi_{n,0}, \xi_n^0]$ is invariant by T^*.

iv) We prove that T^* is a compact continuous operator.
T^* is continuous if and only if $\forall i,\ \psi_i^*$ is continuous where ψ_i^* is the restriction of ψ_i to $[\xi_{1,0}, \xi_1^0] \times ... \times [\xi_{n,0}, \xi_n^0]$.
Let $(\xi_1, ..., \xi_n) \in [\xi_{1,0}, \xi_1^0] \times ... \times [\xi_{n,0}, \xi_n^0]$.
Let $(\xi_1^p, ..., \xi_n^p)_p$ a sequence in $[\xi_{1,0}, \xi_1^0] \times ... \times [\xi_{n,0}, \xi_n^0]$ converging to $(\xi_1, ..., \xi_n)$ for $\|.\|_{(L^2(I\!\!R^N))^n}$.
We have: $\forall 1 \leq i \leq n$,

$$\left\|\frac{\xi_i^p}{1+\epsilon|\xi_i^p|}1_{B_\epsilon} - \frac{\xi_i}{1+\epsilon|\xi_i|}1_{B_\epsilon}\right\|_{L^2(I\!\!R^N)} \leq \frac{1}{\epsilon}\left\|\frac{\epsilon\xi_i^p}{1+\epsilon|\xi_i^p|} - \frac{\epsilon\xi_i}{1+\epsilon|\xi_i|}\right\|_{L^2(I\!\!R^N)}.$$

However the function l defined on $I\!\!R$ by: $\forall x \in I\!\!R,\ l(x) = \frac{x}{1+|x|}$ is Lipschitz and satisfies: $\forall x, y \in I\!\!R,\ |l(x) - l(y)| \leq |x - y|$.
So:

$$\left\|\frac{\xi_i^p}{1+\epsilon|\xi_i^p|} - \frac{\xi_i}{1+\epsilon|\xi_i|}\right\|_{L^2(I\!\!R^N)} \leq \frac{1}{\epsilon}\|\epsilon\xi_i^p - \epsilon\xi_i\|_{L^2(I\!\!R^N)} = \|\xi_i^p - \xi_i\|_{L^2(I\!\!R^N)}.$$

Hence:

$$\frac{\xi_i^p}{1+\epsilon|\xi_i^p|}1_{B_\epsilon} - \frac{\xi_i}{1+\epsilon|\xi_i|}1_{B_\epsilon} \to 0 \text{ in } L^2(I\!\!R^N) \text{ when } p \to +\infty.$$

So ψ_i^* is continuous and therefore $T^*$is a continuous operator.
Moreover, by Proposition 2.4, $(-\Delta + q_i')^{-1}$ is a compact operator.
So T^* is compact.

v) $[\xi_{1,0}, \xi_1^0] \times ... \times [\xi_{n,0}, \xi_n^0]$ is a closed convex subset.
Hence, by the Schauder Fixed Point Theorem, we deduce the existence of $(\xi_1, ..., \xi_n) \in$

$[\xi_{1,0},\xi_1^0]\times ... \times[\xi_{n,0},\xi_n^0]$ such that:
$T^*(\xi_1,...,\xi_n)=(\xi_1,...,\xi_n)$.
$\forall i,\ \xi_i$ depends of ϵ, so we denote: $\xi_i=u_{i,\epsilon}$.
$u_{1,\epsilon},...,u_{n,\epsilon}$ satisfy:

$$(S)\left\{\begin{array}{l}\forall 1\le i\le n\\ (Si)\quad (-\Delta+q_i')u_{i,\epsilon}=m\frac{u_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|}1_{B_\epsilon}+\sum_{j;j\neq i}a_{ij}\frac{u_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|}1_{B_\epsilon}+f_i \text{ in } I\!R^N\,.\end{array}\right.$$

vi) Now we prove that $\forall i,\ (\epsilon u_{i,\epsilon})_\epsilon$ is a bounded sequence in $V_{q_i'}(I\!R^N)$.
Let $\|u\|_{q_i'}=[\int_{I\!R^N}|\nabla u|^2+q_i'|u|^2]^{\frac12}$. We multiply (Si) by $\epsilon^2u_{i,\epsilon}$ and we integrate over $I\!R^N$.
We get:

$$\|\epsilon u_{i,\epsilon}\|^2_{q_i'}\le m\int_{I\!R^N}|\frac{\epsilon u_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|}1_{B_\epsilon}\epsilon u_{i,\epsilon}|+\sum_{j;j\neq i}|a_{ij}|^*\int_{I\!R^N}|\frac{\epsilon u_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|}1_{B_\epsilon}\epsilon u_{i,\epsilon}|$$
$$+\int_{I\!R^N}|\epsilon f_i\epsilon u_{i,\epsilon}|\,.$$

But: $\forall j,\ |\frac{\epsilon u_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|}|<1$.
So there exists a strictly positive constant K such that: $\|\epsilon u_{i,\epsilon}\|^2_{q_i'}\le K\|\epsilon u_{i,\epsilon}\|_{L^2(I\!R^N)}\le K\|\epsilon u_{i,\epsilon}\|_{q_i'}$ and therefore: $\|\epsilon u_{i,\epsilon}\|_{q_i'}\le K$.

vii) We prove now that $\epsilon u_{i,\epsilon}\to 0$ when $\epsilon\to 0$ strongly in $L^2(I\!R^N)$ and weakly in $V_{q_i'}(I\!R^N)$.
We know that the imbedding of $V_{q_i'}(I\!R^N)$ into $L^2(I\!R^N)$ is compact.
The sequence $(\epsilon u_{i,\epsilon})_\epsilon$ is bounded in $V_{q_i'}(I\!R^N)$ so (for a subsequence), we deduce that $\exists u_i^*$ such that:
$\epsilon u_{i,\epsilon}\to u_i^*$ when $\epsilon\to 0$ strongly in $L^2(I\!R^N)$ and weakly in $V_{q_i'}(I\!R^N)$.
Multiplying (Si) by ϵ, we get:

$$(-\Delta+q_i')\epsilon u_{i,\epsilon}=m\frac{\epsilon u_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|}1_{B_\epsilon}+\sum_{j;j\neq i}a_{ij}\frac{\epsilon u_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|}1_{B_\epsilon}+\epsilon f_i \text{ in } I\!R^N.$$

But $\epsilon u_{i,\epsilon}\rightharpoonup u_i^*$ weakly in $V_{q_i}(I\!R^N)$.
So: $\forall\phi\in\mathcal{D}(I\!R^N)$,

$$\int_{I\!R^N}[\nabla(\epsilon u_{i,\epsilon}).\nabla\phi+q_i'\epsilon u_{i,\epsilon}\phi]\to\int_{I\!R^N}[\nabla u_i^*.\nabla\phi+q_i'u_i^*\phi] \text{ when } \epsilon\to 0.$$

Moreover: $\forall\phi\in\mathcal{D}(I\!R^N),\ \int_{I\!R^N}\epsilon f_i\phi\to 0$ when $\epsilon\to 0$.
Moreover we have: $\forall j$

$$\|\frac{\epsilon u_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|}1_{B_\epsilon}-\frac{u_j^*}{1+|u_j^*|}\|^2_{L^2(I\!R^N)}=\int_{B_\epsilon}[\frac{\epsilon u_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|}-\frac{u_j^*}{1+|u_j^*|}]^2$$

$$+\int_{I\!R^N-B_\epsilon}(\frac{u_j^*}{1+|u_j^*|})^2 .$$

Since: $|\frac{u_j^*}{1+|u_j^*|}| \le |u_j^*|$, $\frac{u_j^*}{1+|u_j^*|} \in L^2(I\!R^N)$, hence $\int_{I\!R^N-B_\epsilon}(\frac{u_j^*}{1+|u_j^*|})^2 \to 0$ when $\epsilon \to 0$.
Moreover: $\int_{B_\epsilon}[\frac{\epsilon u_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|} - \frac{u_j^*}{1+|u_j^*|}]^2 \le \int_{I\!R^N}[\frac{\epsilon u_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|} - \frac{u_j^*}{1+|u_j^*|}]^2$

$$\le \|\epsilon u_{j,\epsilon} - u_j^*\|^2_{L^2(I\!R^N)} .$$

But: $\epsilon u_{j,\epsilon} \longrightarrow u_j^*$ when $\epsilon \to 0$ strongly in $L^2(I\!R^N)$.
So: $\frac{\epsilon u_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|}1_{B_\epsilon} \to \frac{u_j^*}{1+|u_j^*|}$ when $\epsilon \to 0$ strongly in $L^2(I\!R^N)$.
Therefore we can pass through the limit and we get:

$$(S')\begin{cases} \forall 1 \le i \le n \\ (S'i) \quad (-\Delta + q_i')u_i^* = m\frac{u_i^*}{1+|u_i^*|} + \sum_{j;j\neq i} a_{ij}\frac{u_j^*}{1+|u_j^*|} \text{ in } I\!R^N . \end{cases}$$

We prove now that for any i, $u_i^* = 0$. We multiply $(S'i)$ by u_i^*, we integrate over $I\!R^N$ and we obtain:

$$\int_{I\!R^N}[|\nabla u_i^*|^2 + q_i'|u_i^*|^2] = \int_{I\!R^N} m\frac{|u_i^*|^2}{1+|u_i^*|} + \sum_{j;j\neq i}\int_{I\!R^N} a_{ij}\frac{u_j^* u_i^*}{1+|u_j^*|}$$

$$\le \int_{I\!R^N} m\frac{|u_i^*|^2}{1+|u_i^*|} + \sum_{j;j\neq i}\int_{I\!R^N} |a_{ij}|^*\frac{|u_j^*||u_i^*|}{1+|u_j^*|}.$$

But: $\forall j$, $\frac{1}{1+|u_j^*|} \le 1$.
So we get:

$$\lambda(q_i')\int_{I\!R^N}|u_i^*|^2 \le m\int_{I\!R^N}|u_i^*|^2 + \sum_{j;j\neq i}|a_{ij}|^*(\int_{I\!R^N}|u_j^*|^2)^{\frac{1}{2}}(\int_{I\!R^N}|u_i^*|^2)^{\frac{1}{2}}.$$

Replacing u_i by u_i^*, we proceed exactly as in lemma 3.1 and we get that $\forall 1 \le i \le n$, $u_i^* = 0$.

viii) We prove now by contradiction that $\forall 1 \le i \le n,\ (u_{i,\epsilon})_\epsilon$ is bounded in $V_{q_i}(I\!R^N)$.
We suppose that: $\exists i_0,\ \|u_{i_0,\epsilon}\|_{q_{i_0}} \to +\infty$ when $\epsilon \to 0$. Let: $\forall 1 \le i \le n$,

$$t_\epsilon = \max_i(\|u_{i,\epsilon}\|_{q_i}) \text{ and } v_{i,\epsilon} = \frac{1}{t_\epsilon}u_{i,\epsilon}.$$

We have $\|v_{i,\epsilon}\|_{q_i} \le 1$ so $(v_{i,\epsilon})_\epsilon$ is a bounded sequence in $V_{q_i}(I\!R^N)$.
Since the imbedding of $V_{q_i}(I\!R^N)$ in $L^2(I\!R^N)$ is compact (Proposition 2.3), there exists v_i such that $v_{i,\epsilon} \to v_i$ when $\epsilon \to 0$ strongly in $L^2(I\!R^N)$ and weakly in $V_{q_i}(I\!R^N)$.
In a weak sense, we have: $\forall 1 \le i \le n$,

$$(-\Delta + q_i')v_{i,\epsilon} = m\frac{v_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|}1_{B_\epsilon} + \sum_{j;j\neq i} a_{ij}\frac{v_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|}1_{B_\epsilon} + \frac{1}{t_\epsilon}f_i \text{ in } I\!R^N.$$

We have: $\forall \phi \in \mathcal{D}(I\!R^N)$,

$$\int_{I\!R^N} [\nabla v_{i,\epsilon}.\nabla \phi + q_i' v_{i,\epsilon}\phi] \to \int_{I\!R^N} [\nabla v_i.\nabla \phi + q_i' v_i \phi] \text{ when } \epsilon \to 0.$$

Moreover $t_\epsilon \to +\infty$ when $\epsilon \to 0$ so:
$\forall \phi \in \mathcal{D}(I\!R^N)$, $\int_{I\!R^N} \frac{1}{t_\epsilon} f_i \phi \to 0$ when $\epsilon \to 0$.
We have also: $\forall 1 \le j \le n$,

$$\|\frac{v_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|} 1_{B_\epsilon} - v_j\|^2_{L^2(I\!R^N)} = \int_{B_\epsilon} [\frac{v_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|} - v_j]^2 + \int_{I\!R^N - B_\epsilon} v_j^2 .$$

But: $v_j \in L^2(I\!R^N)$ so : $\int_{I\!R^N - B_\epsilon} v_j^2 \to 0$ when $\epsilon \to 0$.
Moreover:

$$\int_{B_\epsilon} [\frac{v_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|} - v_j]^2 \le \int_{I\!R^N} [\frac{v_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|} - v_j]^2$$

$$\le 2[\int_{I\!R^N} \frac{(v_{j,\epsilon} - v_j)^2}{(1+\epsilon|u_{j,\epsilon}|)^2} + \int_{I\!R^N} \frac{(\epsilon v_j |u_{j,\epsilon}|)^2}{(1+\epsilon|u_{j,\epsilon}|)^2}] .$$

But: $1+\epsilon|u_{j,\epsilon}| \ge 1$. So: $\int_{I\!R^N} \frac{(v_{j,\epsilon}-v_j)^2}{(1+\epsilon|u_{j,\epsilon}|)^2} \le \int_{I\!R^N} (v_{j,\epsilon} - v_j)^2$.
Since $v_{j,\epsilon} \to v_j$ in $L^2(I\!R^N)$,we get: $\int_{I\!R^N} \frac{(v_{j,\epsilon}-v_j)^2}{(1+\epsilon|u_{j,\epsilon}|)^2} \to 0$ when $\epsilon \to 0$.
Moreover:

$$\frac{(\epsilon v_j |u_{j,\epsilon}|)^2}{(1+\epsilon|u_{j,\epsilon}|)^2} \to 0 \; pp \text{ when } \epsilon \to 0$$

(at least for a subsequence because $\epsilon u_{j,\epsilon} \to 0$ when $\epsilon \to 0$.)
By using the Dominated Convergence Theorem, we deduce that:
$\int_{I\!R^N} \frac{(\epsilon v_j|u_{j,\epsilon}|)^2}{(1+\epsilon|u_{j,\epsilon}|)^2} \to 0$ when $\epsilon \to 0$.
So we can pass through the limit and we get: $\forall 1 \le i \le n$,

$$(-\Delta + q_i')v_i = m v_i + \sum_{j;j\neq i} a_{ij} v_j \text{ in } I\!R^N.$$

By the precedent lemma, we deduce that: $\forall 1 \le i \le n,\ v_i = 0$.
However there exists a sequence (ϵ_n) such that $\exists i_1,\ \|v_{i_1,\epsilon_n}\|_{q_{i_1}} = 1$.
But $v_{i_1,\epsilon_n} \to v_{i_1}$ when $n \to +\infty$. So we get a contradiction.

ix) There exists u_i^0 such that :
$u_{i,\epsilon} \to u_i^0$ strongly in $L^2(I\!R^N)$ and weakly in $V_{q_i}(I\!R^N)$.
We have in a weak sense:

$$(-\Delta + q_i')u_{i,\epsilon} = m\frac{u_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|} 1_{B_\epsilon} + \sum_{j;j\neq i} a_{ij} \frac{u_{j,\epsilon}}{1+\epsilon|u_{j,\epsilon}|} 1_{B_\epsilon} + f_i \text{ in } I\!R^N.$$

But $u_{i,\epsilon} \rightharpoonup u_i^0$ when $\epsilon \to 0$ weakly in $V_{q_i}(I\!\!R^N)$. Hence $\forall \phi \in \mathcal{D}(I\!\!R^N)$,

$$\int_{I\!\!R^N} [\nabla u_{i,\epsilon}.\nabla\phi + q_i' u_{i,\epsilon}\phi] \to \int_{I\!\!R^N} [\nabla u_i^0.\nabla\phi + q_i' u_i^0\phi] \text{ when } \epsilon \to 0.$$

We have also:

$$\|\frac{u_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|} 1_{B_\epsilon} - u_i^0\|^2_{L^2(I\!\!R^N)} = \int_{B_\epsilon} [\frac{u_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|} - u_i^0]^2 + \int_{I\!\!R^N - B_\epsilon} |u_i^0|^2 .$$

By $u_i^0 \in L^2(I\!\!R^N)$ we derive: $\int_{I\!\!R^N - B_\epsilon} |u_i^0|^2 \to 0$ when $\epsilon \to 0$.
Moreover:

$$\begin{aligned}\int_{B_\epsilon} [\frac{u_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|} - u_i^0]^2 &\le \int_{I\!\!R^N} [\frac{u_{i,\epsilon}}{1+\epsilon|u_{i,\epsilon}|} - u_i^0]^2 \\ &\le 2[\int_{I\!\!R^N} \frac{(u_{i,\epsilon}-u_i^0)^2}{(1+\epsilon|u_{i,\epsilon}|)^2} + \int_{I\!\!R^N} \frac{(\epsilon u_i^0|u_{i,\epsilon}|)^2}{(1+\epsilon|u_{i,\epsilon}|)^2}].\end{aligned}$$

Since $1+\epsilon|u_{i,\epsilon}| \ge 1$ we get: $\int_{I\!\!R^N} \frac{(u_{i,\epsilon}-u_i^0)^2}{(1+\epsilon|u_{i,\epsilon}|)^2} \le \int_{I\!\!R^N} (u_{i,\epsilon} - u_i^0)^2$.
But: $u_{i,\epsilon} \to u_i^0$ in $L^2(I\!\!R^N)$. So: $\int_{I\!\!R^N} \frac{(u_{i,\epsilon}-u_i^0)^2}{(1+\epsilon|u_{i,\epsilon}|)^2} \to 0$ when $\epsilon \to 0$.
Moreover:

$$\frac{(\epsilon u_i^0|u_{i,\epsilon}|)^2}{(1+\epsilon|u_{i,\epsilon}|)^2} \to 0 \; pp \text{ when } \epsilon \to 0$$

(at least for a subsequence because $\epsilon u_{i,\epsilon} \to 0$ when $\epsilon \to 0$)
and $\frac{(\epsilon u_i^0|u_{i,\epsilon}|)^2}{(1+\epsilon|u_{i,\epsilon}|)^2} \le |u_i^0|^2$ and $|u_i^0|^2 \in L^1(I\!\!R^N)$.
By using the Dominated Convergence Theorem, we deduce that:
$\int_{I\!\!R^N} \frac{(\epsilon u_i^0|u_{i,\epsilon}|)^2}{(1+\epsilon|u_{i,\epsilon}|)^2} \to 0$ when $\epsilon \to 0$.
So we can pass through the limit and we get: $\forall 1 \le i \le n$,

$$(-\Delta + q_i')u_i^0 = m u_i^0 + \sum_{j;j\ne i} a_{ij} u_j^0 + f_i \text{ in } I\!\!R^N.$$

So we get: $(-\Delta + q_i)u_i^0 = a_{ii}u_i^0 + \sum_{j;j\ne i} a_{ij}u_j^0 + f_i$ in $I\!\!R^N$.
$(u_1^0, ..., u_n^0)$ is a weak solution of (1).

3.2 Study of a limit case

We use again a method in [9]. We rewrite System (1), assuming $\forall 1 \le i \le n,\ q_i = q$:

$$(1) \begin{cases} \text{for } 1 \le i \le n, \\ (1i) \quad L_q u_i := (-\Delta + q)u_i = \sum_{j=1}^n a_{ij}u_j + f_i(x, u_1, ..., u_n) \text{ in } I\!\!R^N . \end{cases}$$

Each a_{ij} is a real constant.
We denote $A = (a_{ij})$ the $n \times n$ matrix, I the $n \times n$ identity matrix, ${}^tU = (u_1...u_n)$ and ${}^tF = (f_1...f_n)$.

Theorem 3.2 *We suppose that $(H1), (H2)$ and $(H3)$ are satisfied.*
We suppose that A has only real eigenvalues. We suppose also that $\lambda(q)$, the principal eigenvalue of $-\Delta + q$, is the largest eigenvalue of A and that it is simple.
Let $X \in I\!R^N$ such that: ${}^tX(\lambda(q)I - A) = 0$.
Then System (1) has a solution if and only if $\int_{I\!R^N} {}^tXF\phi_q = 0$, where ϕ_q is the eigenfunction associated to $\lambda(q)$.

Proof of Theorem 3.2: Let P be a $n \times n$ non singular matrix such that the last line of P is ${}^tX$ and such that $T = PAP^{-1} := (t_{ij})$ where:
$t_{ij} = 0$ if $i > j$; $t_{nn} = \lambda(q)$ and $\forall 1 \leq i \leq n-1,\ t_{ii} < \lambda(q)$.
Let: $W = PU$.
System (1) is equivalent to System (2) : $\quad (-\Delta + q)W = TW + PF$.
Let: ${}^tW = (w_1 ... w_n)$ and $\pi_i = (\delta_{ij})$ where: $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$.
So System (2) is:

$$(2) \begin{cases} \text{for } 1 \leq i \leq n, \\ (2i) \quad L_q w_i := (-\Delta + q)w_i = t_{ii}w_i + \sum_{j; j>i} t_{ij}w_j + \pi_i PF \text{ in } I\!R^N . \end{cases}$$

We have: $(2n) \quad (-\Delta + q)w_n = \lambda(q)w_n +^t XF$ in $I\!R^N$. Equation $(2n)$ has a solution if and only if $\int_{I\!R^N} {}^tXF\phi_q = 0$.
If $\int_{I\!R^N} {}^tXF\phi_q = 0$ is satisfied, first we solve $(2n)$, then we solve $(2.n-1)$ until (2.1) because $\forall 1 \leq i \leq n-1,\ t_{ii} < \lambda(q)$. Then we deduce U (because Matrix P is a non singular matrix).

3.3 Study of a non necessarily cooperative semilinear system of n equations

We rewrite System (1) :

$$(1) \begin{cases} \text{for } 1 \leq i \leq n, \\ (1i) \quad L_{q_i} u_i := (-\Delta + q_i)u_i = \sum_{j=1}^n a_{ij}u_j + f_i(x, u_1, ..., u_n) \text{ in } I\!R^N . \end{cases}$$

We recall $G = (g_{ij})$ the $n \times n$ matrix defined by:
$\forall 1 \leq i \leq n,\ g_{ii} = \lambda(q_i - a_{ii})$ and

$$\forall 1 \leq i, j \leq n,\ i \neq j \Rightarrow g_{ij} = -|a_{ij}|^* \text{ where } |a_{ij}|^* = \sup_{x \in I\!R^N} |a_{ij}(x)|.$$

Let I be the identity matrix.

Theorem 3.3 *We assume that $(H1), (H2)$ and $(H3)$ are satisfied.*
We assume also that Hypothesis $(H4), (H5), (H6)$ are satisfied where

(H4) $\exists s > 0$ *such that $F - sI$ is a non singular M-matrix.*

(H5) $\forall 1 \le i \le n,\ \exists \theta_i \in L^2(I\!R^N), \theta_i > 0,$ *such that :*
$\forall 1 \le i \le n,\ \forall u_1, ..., u_n,\ 0 \le f_i(x, u_1, ..., u_n) \le s u_i + \theta_i;$

(H6) $\forall 1 \le i \le n,\ f_i$ *is Lipschitz for* $(u_1, ..., u_n)$, *uniformly in* x.

Then System (1) *has at least a solution.*

Proof of Thorem 3.3:

a) Construction of an upper and lower solution.

We consider the following system (S):

$$(S)\left\{\begin{array}{l}\forall 1 \le i \le n\,,\\ L_{q_i} u_i := (-\Delta + q_i)u_i = a_{ii}u_i + \sum_{j;j\neq i} |a_{ij}|u_j + s u_i + \theta_i \text{ in } I\!R^N\,.\end{array}\right.$$

By Hypothesis $(H4), (H5)$ we can apply Theorem 2.3.
We deduce the existence of a positive solution $U^0 = (u_1^0, ..., u_n^0)$ in $V_{q_1}(I\!R^N) \times ... \times V_{q_n}(I\!R^N)$ for the system (S). U^0 is an upper solution of System (1).
Let: $U_0 = -U^0 = (-u_1^0, ..., -u_n^0)$.
We have: $\forall 1 \le i \le n,\ (-\Delta + q_i)(-u_i^0) = -(-\Delta + q_i)u_i^0$.
Hence: $(-\Delta + q_i)(-u_i^0) = -a_{ii}u_i^0 - \sum_{j;j\neq i} |a_{ij}|u_j^0 - s u_i^0 - \theta_i$.
So: $\forall 1 \le i \le n$,

$$(-\Delta + q_i)(-u_i^0) \le a_{ii}(-u_i^0) + \sum_{j;j\neq i} a_{ij}(-u_j^0) + f_i(x, -u_1^0, ..., -u_n^0).$$

Therefore U_0 is a lower solution of System (1).

b) Definition of a compact operator.

Let $m \in I\!R^{*+}$ be such that : $\forall 1 \le i \le n,\ m - a_{ii} > 0$.
Let: $q_i' = q_i - a_{ii} + m$.
Let $T : (L^2(I\!R^N))^n \to (L^2(I\!R^N))^n$ defined by: $T(u_1, ..., u_n) = (w_1, ..., w_n)$ such that:

$$(S')\left\{\begin{array}{l}\forall 1 \le i \le n\,,\\ (S'i) \quad (-\Delta + q_i')w_i = m u_i + \sum_{j=1;j\neq i}^{n} a_{ij}u_j + f_i(x, u_1, ..., u_n) \text{ in } I\!R^N\,.\end{array}\right.$$

We prove easily that T is a well defined operator by the scalar case, continuous by $(H6)$ and compact (because $(-\Delta + q_i')^{-1}$ is compact).
We prove now that $T([U_0, U^0]) \subset [U_0, U^0]$.
Let $U = (u_1, ..., u_n) \in [U_0, U^0]$.
We have: $\forall 1 \le i \le n,\ -u_i^0 \le u_i \le u_i^0$.
We have: $(-\Delta + q_i')(u_i^0 - w_i) = m(u_i^0 - u_i) + \sum_{j;j\neq i} |a_{ij}|u_j^0$

$-\sum_{j;j\neq i} a_{ij}u_j + su_i^0 + \theta_i - f_i(x, u_1, ..., u_n)$.
We have: $m(u_i^0 - u_i) \geq 0$.
By $(H5)$, we have: $f_i(x, u_1, ..., u_n) \leq su_i + \theta_i \leq su_i^0 + \theta_i$.
Moreover: $|a_{ij}u_j| \leq |a_{ij}|u_j^0$ so $a_{ij}u_j \leq |a_{ij}|u_j^0$.
So: $(-\Delta + q_i')(u_i^0 - w_i) \geq 0$ and by the scalar case: $u_i^0 - w_i \geq 0$.
In the same way, we have:
$(-\Delta + q_i')(w_i - (-u_i^0)) = m(u_i^0 + u_i) + \sum_{j;j\neq i} |a_{ij}|u_j^0$
$+\sum_{j;j\neq i} a_{ij}u_j + su_i^0 + \theta_i + f_i(x, u_1, ..., u_n)$.
But $-u_i^0 \leq u_i$. So $m(u_i^0 + u_i) \geq 0$. Moreover: $-a_{ij}u_j \leq |a_{ij}|u_j^0$.
By using $(H5)$, we conclude that: $(-\Delta + q_i')(w_i + u_i^0) \geq 0$ and hence: $w_i \geq -u_i^0$. So: $T([U_0, U^0]) \subset [U_0, U^0]$.
$[U_0, U^0]$ is a convex, closed, and bounded subset of $(L^2(I\!R^N))^n$, so by the Schauder Fixed Point Theorem, we deduce that T has a fixed point.
Therefore System (1) has at least a solution.

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