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Characteristic Processes associated with a Discontinuous Martingale

1 Introduction

In [5] the following embedding theorem was proved (cf. [2] for a detailed and complete proof). Let $(M_k)_{k\geq 0}$ be an L^4-martingale on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Then there exists an extension $(\bar\Omega, \bar{\mathcal{F}}, \bar{\mathbf{P}})$ of Ω, a Brownian motion $(B_t)_{t\geq 0}$ relative to a filtration $(\mathcal{G}_t)_{t\geq 0}$ on $\bar\Omega$, and an increasing sequence $(T_n)_{n\geq 1}$ of $(\mathcal{G}_t)$-stopping times, such that the following properties hold:

(i) $M_n = B_{T_n}$ for all $n \geq 1$,

(ii) $\mathbf{E}\{T_n \,|\, M_0, M_1, \cdots\} =$

$$\tfrac{1}{3}\sum_{k=1}^n (M_k - M_{k-1})^2 + \tfrac{2}{3}\sum_{k=1}^n \mathbf{E}\{(M_k - M_{k-1})^2 \,|\, M_0, \cdots, M_{k-1}\},$$

(iii) $\mathrm{Var}\{T_n \,|\, M_0, M_1, \cdots\} =$

$$\begin{aligned} &\tfrac{2}{45}\sum_{k=1}^n (M_k - M_{k-1})^4 + \tfrac{8}{45}\sum_{k=1}^n \mathbf{E}\{(M_k - M_{k-1})^4 \,|\, M_0, \cdots, M_{k-1}\} \\ &+ (\tfrac{4}{9} - c)\sum_{k=1}^n \big(\mathbf{E}\{(M_k - M_{k-1})^2 \,|\, M_0, \cdots, M_{k-1}\}\big)^2 \\ &+ c\sum_{k=1}^n (M_k - M_{k-1})^2 \mathbf{E}\{(M_k - M_{k-1})^2 \,|\, M_0, \cdots, M_{k-1}\} \\ &- c\sum_{k=1}^n (M_k - M_{k-1}) \mathbf{E}\{(M_k - M_{k-1})^3 \,|\, M_0, \cdots, M_{k-1}\}. \end{aligned}$$

The constant $c > 0$ in (iii) depends on the embedding and can be explicitely computed (cf. [2] and [5]).

In [5] a corresponding result is stated (without proof) for continuous time martingales. The roughly outlined idea in [5] is the approximation of a continuous time martingale $(M_t)_{t\geq 0}$ by the discrete time martingales $(M_{\frac{k}{2^m}})_{k\geq 0}$. But this raises the following problems, which are the subject of the present paper.

(A) In which sense do the expressions

$$\sum_{k=1}^{[2^m t]} \mathbf{E}\big\{(M_{\frac{k}{2^m}} - M_{\frac{k-1}{2^m}})^2 \,|M_0, \cdots, M_{\frac{k-1}{2^m}}\big\}$$

up to

$$\sum_{k=1}^{[2^m t]} (M_{\frac{k}{2^m}} - M_{\frac{k-1}{2^m}})\mathbf{E}\big\{(M_{\frac{k}{2^m}} - M_{\frac{k-1}{2^m}})^3 \,|M_0, \cdots, M_{\frac{k-1}{2^m}}\big\}$$

converge for $m \to \infty$?

It seems that convergence only holds under additional assumptions on $(M_t)_{t\geq 0}$. The reason is that the above conditioning is a conditioning relative to a filtration depending on m. If $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ is a given filtration on Ω such that $(M_t)_{t\geq 0}$ is an $\mathbf{F}$-martingale, then the following, slightly changed problem only depends on the fixed filtration $\mathbf{F}$:

(B) In which sense do the expressions

$$\sum_{k=1}^{[2^m t]} \mathbf{E}\big\{(M_{\frac{k}{2^m}} - M_{\frac{k-1}{2^m}})^2 \mid \mathcal{F}_{\frac{k-1}{2^m}}\big\}$$

up to

$$\sum_{k=1}^{[2^m t]} (M_{\frac{k}{2^m}} - M_{\frac{k-1}{2^m}})\mathbf{E}\big\{(M_{\frac{k}{2^m}} - M_{\frac{k-1}{2^m}})^3 \mid \mathcal{F}_{\frac{k-1}{2^m}}\big\}$$

converge for $m \to \infty$?

It turns out that (B) has a quite general solution. Convergence always takes place for the topology $\sigma(L^1, L^\infty)$. An inspection of the proof of the embedding theorem for continuous time martingales in [2] shows that this $\sigma(L^1, L^\infty)$-convergence is sufficient.

2 Statement of the Problem

Let Π denote the set of all partitions $\pi = (0 = t_0 < t_1 < \cdots)$ of $\mathbf{R}_+$ for which $\lim_{k\to\infty} t_k = \infty$ and $|\pi| := \sup_{k\geq 1}(t_k - t_{k-1}) < \infty$. We will tacitely identify such a partition π with its associated point set $\{t_k | k \geq 0\}$. A sequence $(\pi_n)_{n\geq 1}$ in Π will be called a *null-sequence of partitions*, if $\pi_m \subset \pi_n$ for $m \leq n$ and $\lim_{n\to\infty} |\pi_n| = 0$.

Now suppose that $M = (M_t)_{t\geq 0}$ is a right continuous martingale relative to a filtration $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$. More shortly, we will also say that $(M, \mathbf{F})$ is a martingale. $(M, \mathbf{F})$ is called a *discrete time martingale*, if there exists a partition $\pi = (0 = t_0 < t_1 < \cdots) \in \Pi$ such that $M_t = M_{t_k}$ and $\mathcal{F}_t = \mathcal{F}_{t_k}$ for $t_k \leq t < t_{k+1}$ and $k \geq 0$. More shortly, we will also say that $((M_{t_k}), (\mathcal{F}_{t_k}))$ is a discrete time martingale. If $(M, \mathbf{F})$ is a general right continuous martingale and $\pi = (0 = t_0 < t_1 < \cdots) \in \Pi$, and if we set $\pi(M) = (M_{t_k})_{k \geq 0}$ and $\pi(\mathbf{F}) = (\mathcal{F}_{t_k})_{k \geq 0}$, then clearly $(\pi(M), \pi(\mathbf{F}))$ is a discrete time martingale. Moreover, for every filtration $\mathbf{H} = (\mathcal{H}_{t_k})_{k \geq 0}$ such that

$$\sigma(M_{t_0}, \cdots, M_{t_k}) \subset \mathcal{H}_{t_k} \subset \mathcal{F}_{t_k}$$

for all $k \geq 0$, also $(\pi(M), \mathbf{H})$ is a discrete time martingale.

If $(\pi_n)_{n \geq 1}$ is a null-sequence of partitions, where $\pi_n = (0 = t_0^n < t_1^n < \cdots)$, then we will call a sequence $(\mathbf{H}^n)_{n \geq 1}$ of discrete filtrations $\mathbf{H}^n = (\mathcal{H}^n_{t^n_k})_{k \geq 0}$ an *approximating sequence of filtrations* for the given martingale $(M, \mathbf{F})$, if (i) M_t is $\mathcal{H}^n_t$-measurable for every $t \in \pi_n$ and $n \geq 1$, (ii) $\mathcal{H}^m_t \subset \mathcal{H}^n_t$ for $t \in \pi_m$ and $m \leq n$, and (iii) $\mathcal{F}_{t-} \subset \bigvee_{n \geq 1} \mathcal{H}^n_t \subset \mathcal{F}_t$ for all $t \geq 0$.

Motivated by the embedding result stated in the introduction, we give the following definition for discrete time martingales.

Definition 2.1 *Let $(M, \mathbf{F}) = ((M_{t_k}), (\mathcal{F}_{t_k}))$ be a given discrete time martingale.*
(1) *If $(M, \mathbf{F})$ is square integrable, then the processes $(M, \mathbf{F})^{(2)}$ and $(M, \mathbf{F})^{(\tilde{2})}$, defined by*

$$\begin{aligned}
(M, \mathbf{F})^{(2)}_t &:= \sum_{k \geq 1} 1_{[0,t]}(t_k)(M_{t_k} - M_{t_{k-1}})^2\,, \ \textit{and} \\
(M, \mathbf{F})^{(\tilde{2})}_t &:= \sum_{k \geq 1} 1_{[0,t]}(t_k)\mathbf{E}\big\{(M_{t_k} - M_{t_{k-1}})^2 \mid \mathcal{F}_{t_{k-1}}\big\}\,,
\end{aligned}$$

will be called the first order characteristics *of $(M, \mathbf{F})$.*
(2) *If $(M, \mathbf{F})$ is in addition even an L^4-martingale, then the processes*

$$(M, \mathbf{F})^{(4)},\ (M, \mathbf{F})^{(\tilde{4})},\ (M, \mathbf{F})^{(2,\tilde{2})},\ (M, \mathbf{F})^{(\tilde{2},\tilde{2})},\ (M, \mathbf{F})^{(1,\tilde{3})}\,,$$

defined by

$$\begin{aligned}
(M, \mathbf{F})^{(4)}_t &:= \sum_{k \geq 1} 1_{[0,t]}(t_k)(M_{t_k} - M_{t_{k-1}})^4\,, \\
(M, \mathbf{F})^{(\tilde{4})}_t &:= \sum_{k \geq 1} 1_{[0,t]}(t_k)\mathbf{E}\big\{(M_{t_k} - M_{t_{k-1}})^4 \mid \mathcal{F}_{t_{k-1}}\big\}\,, \\
(M, \mathbf{F})^{(2,\tilde{2})}_t &:= \sum_{k \geq 1} 1_{[0,t]}(t_k)(M_{t_k} - M_{t_{k-1}})^2\mathbf{E}\big\{(M_{t_k} - M_{t_{k-1}})^2 \mid \mathcal{F}_{t_{k-1}}\big\}\,, \\
(M, \mathbf{F})^{(\tilde{2},\tilde{2})}_t &:= \sum_{k \geq 1} 1_{[0,t]}(t_k)\big(\mathbf{E}\big\{(M_{t_k} - M_{t_{k-1}})^2 \mid \mathcal{F}_{t_{k-1}}\big\}\big)^2\,, \ \textit{and} \\
(M, \mathbf{F})^{(1,\tilde{3})}_t &:= \sum_{k \geq 1} 1_{[0,t]}(t_k)(M_{t_k} - M_{t_{k-1}})\mathbf{E}\big\{(M_{t_k} - M_{t_{k-1}})^3 \mid \mathcal{F}_{t_{k-1}}\big\}\,,
\end{aligned}$$

are called the second order characteristics *of* $(M, \mathbf{F})$.

Now we can formulate more precisely the problem which is the subject of the present paper: Let (π_n) be a fixed null-sequence in Π and $(M, \mathbf{F})$ be a given right continuous L^4-martingale. Suppose that $(\mathbf{H}^n)_{n\geq 1}$ is an approximating sequence of filtrations for $(M, \mathbf{F})$. Does there exists a "reasonable" topology, for which the processes $(\pi_n(M), \mathbf{H}^n)^{(2)}$, $(\pi_n(M), \mathbf{H}^n)^{(\tilde{2})}$, $(\pi_n(M), \mathbf{H}^n)^{(4)}$, $(\pi_n(M), \mathbf{H}^n)^{(\tilde{4})}$, $(\pi_n(M), \mathbf{H}^n)^{(2,\tilde{2})}$, $(\pi_n(M), \mathbf{H}^n)^{(\tilde{2},\tilde{2})}$, and $(\pi_n(M), \mathbf{H}^n)^{(1,\tilde{3})}$ converge for $n \to \infty$?

In case that $\mathbf{H}^n = \pi_n(\mathbf{F})$ for all $n \geq 1$, we will simply write $(\pi_n(M))^{(2)}$, $(\pi_n(M))^{\tilde{2}}$ etc. instead of $(\pi_n(M), \pi_n(\mathbf{F}))^{(2)}$, $(\pi_n(M), \pi_n(\mathbf{F}))^{\tilde{2}}$ etc.. Then the main result of this paper is the following theorem.

Theorem 2.2 *The limits*

$$
\begin{aligned}
(M)_t^{(2)} &:= \lim_{n\to\infty} (\pi_n(M))_t^{(2)}, &\quad (M)_t^{(\tilde{2})} &:= \lim_{n\to\infty} (\pi_n(M))_t^{\tilde{2}}, \\
(M)_t^{(4)} &:= \lim_{n\to\infty} (\pi_n(M))_t^{(4)}, &\quad (M)_t^{(\tilde{4})} &:= \lim_{n\to\infty} (\pi_n(M))_t^{(\tilde{4})}, \\
(M)_t^{(2,\tilde{2})} &:= \lim_{n\to\infty} (\pi_n(M))_t^{(2,\tilde{2})}, &\quad (M)_t^{(\tilde{2},\tilde{2})} &:= \lim_{n\to\infty} (\pi_n(M))_t^{(\tilde{2},\tilde{2})}, \ \textit{and} \\
(M)_t^{(1,\tilde{3})} &:= \lim_{n\to\infty} (\pi_n(M))_t^{(1,\tilde{3})}
\end{aligned}
$$

exist for all $t \geq 0$ *for the topology* $\sigma(L^1, L^\infty)$.

Definition 2.3 *The limit processes* $(M)^{(2)}$ *and* $(M)^{(\tilde{2})}$ *are again called the* first order characteristics *of* $(M, \mathbf{F})$, *and the limit processes* $(M)^{(4)}$, $(M)^{(\tilde{4})}$, $(M)^{(2,\tilde{2})}$, $(M)^{(\tilde{2},\tilde{2})}$, $(M)^{(1,\tilde{3})}$ *are called the* second order characteristics *of* $(M, \mathbf{F})$.

The first order characteristics $(M)^{(2)}$ and $(M)^{(\tilde{2})}$ are of course just the optional and the predictable quadratic variation $[M]$ and $<M>$, and we will use later also these more usual notations.

In the next section we present first some known results on the first order characteristics and give in that case also solutions for the above stated general approximation problem. In the last section we will finally prove theorem 2.2.

3 Compensators of increasing Processes

The existence of $[M] = (M)^{(2)}$ for the L^1-norm is well known and will not be discussed here. Since $<M> = (M)^{(\tilde{2})}$ is the so-called compensator of the increasing process $[M]$, we first recall some known results on increasing processes and their compensators.

Suppose that $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$ is a fixed standard filtration on Ω, i.e. $\mathbf{F}$ is assumed to be right continuous and every $\mathcal{F}_t$ is assumed to contain all $\mathbf{P}$-null sets of the $\mathbf{P}$-completion of $\mathcal{F}_\infty := \bigvee_{t\geq 0} \mathcal{F}_t$. Let $X = (X_t)_{t\geq 0}$ be a given increasing, right continuous, $\mathbf{F}$-adapted process, and denote by $\tilde{X} = (\tilde{X}_t)_{t\geq 0}$ the *compensator* of X (also called the dual predictable projection of X). $\tilde{X}$ is the (up to $\mathbf{P}$-equality) unique increasing, right continuous, $\mathbf{F}$-predictable process with $\tilde{X}_0 = 0$ such that $X - \tilde{X}$ is a local martingale (see e.g. [1], theorem 15.2 for the existence and uniqueness of $\tilde{X}$).

The process X has the following general structure (cf. [1], ch.VI): there exist
(i) an increasing, continuous, $\mathbf{F}$-adapted process X^c, and
(ii) sequences $(S_n)_{n\geq 1}$, $(T_n)_{n\geq 1}$ of predictable resp. totally inaccessible stopping times with pairwise disjoint graphs,
such that

$$X \;=\; X^c + \sum_{n\geq 1} \triangle X_{S_n}\, 1_{[S_n,\infty[} \;+\; \sum_{n\geq 1} \triangle X_{T_n}\, 1_{[T_n,\infty[}\,.$$

Let G_n $(n \geq 1)$ denote the (necessarily continuous) compensator of the jump process $\triangle X_{T_n}\, 1_{[T_n,\infty[}$. Then the compensator $\tilde{X}$ of X has the structure

$$\tilde{X} \;=\; \tilde{X^c} \;+\; \sum_{n\geq 1} \mathbf{E}\big\{\triangle X_{S_n} \,|\, \mathcal{F}_{S_n -}\big\}\, 1_{[S_n,\infty[}\,,$$

with $\tilde{X^c} = X^c + \sum_{n\geq 1} G_n$. As a consequence, one has the following equivalent assertions:
(a) X is regular (or quasi-left-continuous),
(b) X has no jumps at predictable stopping times, and
(c) $\tilde{X}$ is continuous.

The following result on the topological relations between increasing processes and their compensators can be found in [1] (ch.VII, Th.18 and 20).

Theorem 3.1 *Suppose that $(X^n)_{n\geq 0}$ is a sequence of right continuous, increasing, $\mathbf{F}$-adapted processes, which belongs uniformly to class (D). If for every stopping time T*

$$\lim_{n\to\infty} X^n_T \;=\; X^0_T \quad for \;\; \sigma(L^1, L^\infty)\,,$$

then also

$$\lim_{n\to\infty} \widetilde{X^n_T} \;=\; \widetilde{X^0_T} \quad for \;\; \sigma(L^1, L^\infty)\,,$$

where $(\widetilde{X^n})_{n\geq 0}$ denotes the associated sequence of compensators. If $(X^n)_{n\geq 0}$ is a sequence such that X^0 is regular (so that $\widetilde{X^0}$ is continuous), and if $(X^n)_{n\geq 1}$ is increasing with limit X^0, then even

$$\lim_{n\to\infty} \widetilde{X^n_T} \;=\; \widetilde{X^0_T}$$

for the L^1-norm.

As a consequence of the above theorem one gets that the compensator of an increasing process can be obtained as the limit of compensators of discrete increasing processes (cf. [1], Th.21). We give a different proof than in [1], which seems to be more elementary. Moreover, our proof can serve at the same time as an existence proof for the compensator.

Theorem 3.2 *Let $X = (X_t)_{t\geq 0}$ be a given increasing, right continuous, integrable and $\mathbf{F}$-adapted process. Let further $(\pi_n)_{n\geq 1}$ be a null-sequence of partitions of $\mathbf{R}_+$, and define for every $n \geq 1$ and $t \geq 0$*

$$C^n(X)_t \;:=\; \sum_{k\geq 1} 1_{[o,t]}(t_k^n)\mathbf{E}\{X_{t_k^n} - X_{t_{k-1}^n} \,|\, \mathcal{F}_{t_{k-1}^n}\}\,.$$

Then there exists an increasing, right continuous, integrable and predictable process $C(X)$ such that

$$\lim_{n\to\infty} C^n(X)_t \;=\; C(X)_t \quad for \quad \sigma(L^1, L^\infty)$$

for all $t \geq 0$, and such that $C(X)$ is the compensator $\tilde{X}$ of X.

Proof: Let $Z \in L^\infty(\Omega)$ be given and suppose that $(Y_t)_{t\geq 0}$ is a cadlag-modification of $(\mathbf{E}\{Z|\mathcal{F}_t\})_{t\geq 0}$. Then we get for every fixed $t \geq 0$

$$\begin{aligned}
&\mathbf{E}\{Z\, C^n(X)_t\} \\
&= \mathbf{E}\big\{Z \sum_{k\geq 1} 1_{[o,t]}(t_k^n)\mathbf{E}\{X_{t_k^n} - X_{t_{k-1}^n} \,|\, \mathcal{F}_{t_{k-1}^n}\}\big\} \\
&= \sum_{k\geq 1} 1_{[o,t]}(t_k^n)\mathbf{E}\big\{\, Z\, \mathbf{E}\{X_{t_k^n} - X_{t_{k-1}^n} \,|\, \mathcal{F}_{t_{k-1}^n}\}\big\} \\
&= \sum_{k\geq 1} 1_{[o,t]}(t_k^n)\mathbf{E}\big\{\mathbf{E}\{Z|\mathcal{F}_{t_{k-1}^n}\}(X_{t_k^n} - X_{t_{k-1}^n})\big\} \\
&= \sum_{k\geq 1} 1_{[o,t]}(t_k^n)\mathbf{E}\big\{Y_{t_{k-1}^n}(X_{t_k^n} - X_{t_{k-1}^n})\big\} \\
&= \mathbf{E}\big\{\sum_{k\geq 1} 1_{[o,t]}(t_k^n)\, Y_{t_{k-1}^n}(X_{t_k^n} - X_{t_{k-1}^n})\big\}\,.
\end{aligned}$$

If we define

$$F_n \;:=\; \sum_{k\geq 1} 1_{[o,t]}(t_k^n) Y_{t_{k-1}^n}\, 1_{]t_{k-1}^n, t_k^n]}\,,$$

the above equation just reads

$$\mathbf{E}\{Z\, C^n(X)_t\} \;=\; \mathbf{E}\big\{\int_0^t F_n(s)\, dX_s\big\}\,.$$

By the definition of F_n and Y we have $\lim_{n\to\infty} F_n(s) = Y_{s-}$ for every $s > 0$, and Lebesgue's theorem yields

$$\lim_{n\to\infty} \mathbf{E}\{Z\, C^n(X)_t\} \;=\; \mathbf{E}\big\{\int_0^t Y_{s-}\, dX_s\big\}\,.$$

Especially, this implies that $(C^n(X)_t)_{n\geq 1}$ is a $\sigma(L^1, L^\infty)$-Cauchy sequence. Because of

$$\mathbf{E}\, C^n(X)_t \;\leq\; \mathbf{E}\, X_t \;=:\; r_t$$

we have that $(C^n(X)_t)_{n\geq 1}$ is contained in the L^1-ball $\{X \in L^1 |\, \mathbf{E}|X| \leq r_t\}$, which is weakly sequentially complete (cf. e.g. [6], p.121). This now implies that there exists a $\bar{C}(X)_t \in L^1$ such that

$$\lim_{n\to\infty} \mathbf{E}\{Z\, C^n(X)_t\} \;=\; \mathbf{E}\{Z\, \bar{C}(X)_t\}$$

for all $Z \in L^\infty$.

Now consider the so obtained process $(\bar{C}(X)_t)_{t\geq 0}$. It is easy to see that $(\bar{C}(X)_t)_{t\geq 0}$ is $\mathbf{F}$-adapted and that

$$\bar{C}(X)_s \;\leq\; \bar{C}(X)_t$$

$\mathbf{P}$-a.s. for all $s \leq t$. Clearly, $(\bar{C}(X)_t)_{t\geq 0}$ is in general not necessarily right continuous. So let us show that $(\bar{C}(X)_t)_{t\geq 0}$ has a right continuous, increasing modification.

We choose a null set N such that for all $\omega \notin N$ and all $s, t \in \mathbf{Q}_+$ with $s \leq t$ we have

$$\bar{C}(X)_s(\omega) \;\leq\; \bar{C}(X)_t(\omega)\,.$$

Then we define

$$C(X)_t \;:=\; 1_{N^c} \inf_{u>t, u\in\mathbf{Q}} \bar{C}(X)_u\,.$$

It follows from the properties of $\mathbf{F}$ that the process $(C(X)_t)_{t\geq 0}$ is an $\mathbf{F}$-adapted, increasing and right continuous process. Furthermore,

$$\begin{aligned}
\mathbf{E}\{Z\, C(X)_t\} &= \inf\nolimits_{u>t, u\in\mathbf{Q}} \mathbf{E}\{Z\, \bar{C}(X)_u\} &&= \inf_{u>t, u\in\mathbf{Q}} \mathbf{E}\{\int_0^u Y_{s-}\, dX_s\} \\
&= \mathbf{E}\{\textstyle\int_0^t Y_{s-}\, dX_s\} &&= \mathbf{E}\{Z\, \bar{C}(X)_t\}
\end{aligned}$$

for all $t \geq 0$ and all $Z \in L^\infty$ with $Z \geq 0$. It follows that

$$\mathbf{P}\{\, C(X)_t \;=\; \bar{C}(X)_t \,\} \;=\; 1$$

for all $t \geq 0$, i.e. $(C(X)_t)_{t\geq 0}$ is a modification of $(\bar{C}(X)_t)_{t\geq 0}$.

Altogether we have proved up to now that there exists an increasing, right continuous, integrable and $\mathbf{F}$-adapted process $(C(X)_t)_{t\geq 0}$ such that

$$\lim_{n\to\infty} \mathbf{E}\{Z\, C^n(X)_t\} \;=\; \mathbf{E}\{Z\, C(X)_t\}$$

for all $t \geq 0$ and all $Z \in L^\infty$.

It is an easy exercise to show that $X - C(X)$ is an $\mathbf{F}$-martingale. This follows from the $\sigma(L^1, L^\infty)$-convergence and the observation that for every $n \geq 1$ the process

$(X_{t_k^n} - C^n(X)_{t_k^n})_{k\geq 0}$ is a martingale relative to the filtration $(\mathcal{F}_{t_k^n})_{k\geq 0}$. Hence for the proof of the theorem it remains to show that $C(X)$ is predictable. We do this by proving the equivalent assertion that $C(X)$ is a natural process. This means that we have to show that

$$\mathbf{E}\{Y_t\, C(X)_t\} \;=\; \mathbf{E}\{\int_0^t Y_{s-}\, dC(X)_s\}$$

holds for every $t \geq 0$ and every non-negative, bounded cadlag $\mathbf{F}$-martingale $(Y_t)_{t\geq 0}$.

The idea is to apply to $C(X)$ the same procedure which we applied to X. For $n \geq m > 1$ we have the following simple relation:

$$\begin{aligned}
&C^m(C^n(X))_t \\
&\quad= \sum_{j\geq 1} 1_{[0,t]}(t_j^m)\mathbf{E}\big\{C^n(X)_{t_j^m} - C^n(X)_{t_{j-1}^m} \,|\, \mathcal{F}_{t_{j-1}^m}\big\} \\
&\quad= \sum_{j\geq 1} 1_{[0,t]}(t_j^m)\mathbf{E}\Big\{\sum_{k\in I_j^m} \mathbf{E}\big\{X_{t_k^n} - X_{t_{k-1}^n} \,|\, \mathcal{F}_{t_{k-1}^n}\big\} \,|\, \mathcal{F}_{t_{j-1}^m}\Big\} \\
&\qquad\qquad\qquad\qquad (I_j^m \;:=\; \{k | t_{j-1}^m < t_k^n \leq t_j^m\}) \\
&\quad= \sum_{j\geq 1} 1_{[0,t]}(t_j^m) \sum_{k\in I_j^m} \mathbf{E}\big\{X_{t_k^n} - X_{t_{k-1}^n} \,|\, \mathcal{F}_{t_{j-1}^m}\big\} \\
&\quad= \sum_{j\geq 1} 1_{[0,t]}(t_j^m)\mathbf{E}\big\{X_{t_j^m} - X_{t_{j-1}^m} \,|\, \mathcal{F}_{t_{j-1}^m}\big\} \\
&\quad= C^m(X)_t\,.
\end{aligned}$$

Now let $(Y_t)_{t\geq 0}$ be a given non-negative, bounded $\mathbf{F}$-cadlag-martingale. Then we have for every $t \geq 0$ on one side

(i) $\qquad \lim_{m\to\infty} \mathbf{E}\{Y_t\, C^m(X)_t\} \;=\; \mathbf{E}\{Y_t\, C(X)_t\}\,,$

and on the other side we get

$$\begin{aligned}
&\mathbf{E}\{Y_t\, C^m(X)_t\} \;=\; \mathbf{E}\{Y_t\, C^m(C^n(X))_t\} \quad (n \geq m) \\
&\quad= \mathbf{E}\Big\{Y_t \sum_{j\geq 1} 1_{[0,t]}(t_j^m)\mathbf{E}\big\{C^n(X)_{t_j^m} - C^n(X)_{t_{j-1}^m} \,|\, \mathcal{F}_{t_{j-1}^m}\big\}\Big\} \\
&\quad= \mathbf{E}\Big\{\sum_{j\geq 1} 1_{[0,t]}(t_j^m) Y_{t_{j-1}^m}\big(C^n(X)_{t_j^m} - C^n(X)_{t_{j-1}^m}\big)\Big\}\,,
\end{aligned}$$

which gives for $n \to \infty$

$$\mathbf{E}\{Y_t\, C^m(X)_t\} \;=\; \mathbf{E}\Big\{\sum_{j\geq 1} 1_{[0,t]}(t_j^m) Y_{t_{j-1}^m}\big(C(X)_{t_j^m} - C(X)_{t_{j-1}^m}\big)\Big\}\,,$$

and hence

(ii) $\qquad \lim_{m\to\infty} \mathbf{E}\{Y_t\, C^m(X)_t\} \;=\; \mathbf{E}\{\int_0^t Y_{s-}\, dC(X)_s\}\,,$

and we have proved that $C(X)$ is natural. $\square$

Remark 3.3 If X is continuous, then one can prove that $\lim_{n\to\infty} C^n(X)_t = C(X)_t = X_t$ even for the L^1-norm. But in general this is not true. There is a counterexample of Dellacherie and Doléans (cf. [1] for a reference).

Let $(\mathbf{H}^n)_{n\geq 1}$ be an increasing sequence of filtrations $\mathbf{H}^n = (\mathcal{H}_t^n)_{t\geq 0}$. Define $\mathbf{H}^\infty$ by $\mathcal{H}_t^\infty := \bigvee_{n\geq 1} \mathcal{H}_t^n$ for $t \geq 0$ and let $\mathbf{F}$ be the standard filtration generated by $\mathbf{H}^\infty$. In analogy to theorem 3.2 one could ask the following question (which is related to the general approximation problem stated in section 2):

Let $X = (X_t)_{t\geq 0}$ be an increasing, right continuous, $\mathbf{F}$-adapted process. Define for every $n \geq 1$

$$C^n(X)_t := \sum_{k\geq 1} 1_{[0,t]}(t_k^n)\mathbf{E}\{X_{t_k^n} - X_{t_{k-1}^n} \mid \mathcal{H}^n_{t^n_{k-1}}\}.$$

Is it again true that

$$\lim_{n\to\infty} C^n(X)_t = \tilde{X}_t$$

for every $t \geq 0$ for the topology $\sigma(L^1, L^\infty)$?

Looking at the proof of theorem (3.2), there is the following problem. If $Z \in L^\infty$ is given, and if the function F_n is now defined as

$$F_n = \sum_{k\geq 1} 1_{[0,t]}(t_k^n)\mathbf{E}\{Z \mid \mathcal{H}^n_{t^n_{k-1}}\} 1_{]t^n_{k-1}, t^n_k]},$$

then clearly

$$F_n(s) \longrightarrow Y_{s-} := \mathbf{E}\{Z|\mathcal{F}_{s-}\}$$

$\mathbf{P}$-a.s. (!) for every $s \geq 0$. But it is not at all clear (and probably not true) that outside *one* fixed null-set one has the convergence $F_n(s) \to Y_{s-}$ for all $s \geq 0$. The reason is a kind of regularity problem for two-parameter martingales: Consider

$$\left(\mathbf{E}\{Z|\mathcal{H}_t^n\}\right)_{1\leq n\leq\infty, t\geq 0}.$$

This is a two-parameter martingale. Does there exist a modification $(Y_{n,t})_{1\leq n\leq\infty, t\geq 0}$, such that

$$\lim_{n\to\infty, s\nearrow t} Y_{n,s} = Y_{\infty,t}$$

for all $t \geq 0$?

The following result indicates that even in case that X is a continuous process the situation differs from the situation, where just one filtration $\mathbf{F}$ is involved.

Theorem 3.4 *Let $(\pi_n)_{n\geq 1}$ be a null-sequence of partitions and $\mathbf{H}^n$ $(n \geq 1)$, $\mathbf{F}$ filtrations as above. If X is an increasing, continuous, $\mathbf{F}$-adapted process, and if*

$$C^n(X)_t := \sum_{k\geq 1} 1_{[0,t]}(t_k^n)\mathbf{E}\{X_{t_k^n} - X_{t_{k-1}^n} \mid \mathcal{H}^n_{t^n_{k-1}}\}$$

for $n \geq 1$ and $t \geq 0$, then for every $t \geq 0$

$$\lim_{n\to\infty} C^n(X)_t \;=\; X_t$$

for the topolgy $\sigma(L^1, L^\infty)$.

Proof: (1) For every integrable process $Y = (Y_t)_{t\geq 0}$ we set

$$C^n(X)_t \;:=\; \sum_{k\geq 1} 1_{[0,t]}(t^n_k)\mathbf{E}\big\{Y_{t^n_k} - Y_{t^n_{k-1}} \,|\, \mathcal{H}^n_{t^n_{k-1}}\big\}\,.$$

Now assume that Y is even square integrable. Then we will first prove an elementary inequality for $\mathbf{E}\{C^n(Y)^2_t\}$. For a shorter notation we set

$$m(t) := m(t,n) := \max\{k \geq 1 |\, t^n_k \leq t\}\,.$$

Then we get

$$\begin{aligned}
\mathbf{E}\{C^n(Y)^2_t\} \;&=\; \mathbf{E}\big(\sum_{k=1}^{m(t)}\mathbf{E}\big\{Y_{t^n_k} - Y_{t^n_{k-1}} \,|\, \mathcal{H}^n_{t^n_{k-1}}\big\}\big)^2 \\
&=\; \mathbf{E}\sum_{k=1}^{m(t)}\big(\mathbf{E}\big\{Y_{t^n_k} - Y_{t^n_{k-1}} \,|\, \mathcal{H}^n_{t^n_{k-1}}\big\}\big)^2 \\
&\quad + 2\sum_{k<j}\mathbf{E}\big(\mathbf{E}\big\{Y_{t^n_k} - Y_{t^n_{k-1}} \,|\, \mathcal{H}^n_{t^n_{k-1}}\big\}\mathbf{E}\big\{Y_{t^n_j} - Y_{t^n_{j-1}} \,|\, \mathcal{H}^n_{t^n_{j-1}}\big\}\big) \\
&=\; \mathbf{E}\sum_{k=1}^{m(t)}\big(\mathbf{E}\big\{Y_{t^n_k} - Y_{t^n_{k-1}} \,|\, \mathcal{H}^n_{t^n_{k-1}}\big\}\big)^2 \\
&\quad + 2\sum_{k<j}\mathbf{E}\big(\mathbf{E}\big\{Y_{t^n_k} - Y_{t^n_{k-1}} \,|\, \mathcal{H}^n_{t^n_{k-1}}\big\}\mathbf{E}\big\{Y_{t^n_j} - Y_{t^n_{j-1}} \,|\, \mathcal{H}^n_{t^n_{k-1}}\big\}\big) \\
&\leq\; 2\sum_{k\leq j}\mathbf{E}\big(\mathbf{E}\big\{Y_{t^n_k} - Y_{t^n_{k-1}} \,|\, \mathcal{H}^n_{t^n_{k-1}}\big\}\mathbf{E}\big\{Y_{t^n_j} - Y_{t^n_{j-1}} \,|\, \mathcal{H}^n_{t^n_{k-1}}\big\}\big) \\
&=\; 2\sum_{k=1}^{m(t)}\mathbf{E}\big(\mathbf{E}\big\{Y_{t^n_k} - Y_{t^n_{k-1}} \,|\, \mathcal{H}^n_{t^n_{k-1}}\big\}\mathbf{E}\big\{Y_{t^n_{m(t)}} - Y_{t^n_{k-1}} \,|\, \mathcal{H}^n_{t^n_{k-1}}\big\}\big) \\
&=\; 2\sum_{k=1}^{m(t)}\mathbf{E}\big((Y_{t^n_{m(t)}} - Y_{t^n_{k-1}})\mathbf{E}\big\{Y_{t^n_k} - Y_{t^n_{k-1}} \,|\, \mathcal{H}^n_{t^n_{k-1}}\big\}\big)\;\;.
\end{aligned}$$

and we have obtained the inequality

$$\mathbf{E}\{C^n(Y)^2_t\} \leq 2\mathbf{E}\Big(\max_{0\leq k<m(t)}|Y_{t^n_{m(t)}} - Y_{t^n_k}|\sum_{k=1}^{m(t)}\big|\mathbf{E}\big\{Y_{t^n_k} - Y_{t^n_{k-1}} \,|\, \mathcal{H}^n_{t^n_{k-1}}\big\}\big|\Big)\,. \tag{3.4.1}$$

If Y is increasing we have especially

$$\begin{aligned}\mathbf{E}\{C^n(Y)_t^2\} &\leq 2\,\mathbf{E}\big((Y_t - Y_0)\,C^n(Y)_t\big)\\ &\leq 2\left(\mathbf{E}(Y_t - Y_0)^2\right)^{\frac{1}{2}}\left(\mathbf{E}(C^n(Y)_t^2)\right)^{\frac{1}{2}},\end{aligned}$$

which implies

$$\mathbf{E}\{C^n(Y)_t^2\} \leq 4\,\mathbf{E}(Y_t - Y_0)^2\,. \tag{3.4.2}$$

Now suppose that $Z = X - Y$, where X and Y are increasing and square integrable processes. Then we get from (3.4.1)

$$\begin{aligned}&\mathbf{E}\{C^n(Y)_t^2\}\\ &\quad\leq 2\,\mathbf{E}\big(\max_{0\leq k<m(t)} |Z_{t^n_{m(t)}} - Z_{t^n_k}|\,(C^n(X)_t + C^n(Y)_t)\big)\\ &\quad\leq 2\left(\mathbf{E}\big\{\max_{0\leq k<m(t)}(Z_{t^n_{m(t)}} - Z_{t^n_k})^2\big\}\right)^{\frac{1}{2}}\left(\mathbf{E}\big\{C^n(X)_t + C^n(Y)_t\big\}^2\right)^{\frac{1}{2}},\end{aligned}$$

and (3.4.2) gives

$$\begin{aligned}&\mathbf{E}\{C^n(Y)_t^2\}\\ &\quad\leq 4\left(\mathbf{E}\big\{\max_{0\leq k<m(t)}(Z_{t^n_{m(t)}} - Z_{t^n_k})^2\big\}\right)^{\frac{1}{2}}\Big(\left(\mathbf{E}(X_t - X_0)^2\right)^{\frac{1}{2}} + \left(\mathbf{E}(Y_t - Y_0)^2\right)^{\frac{1}{2}}\Big)\,.\end{aligned} \tag{3.4.3}$$

This inequality will be used in the next step of the proof.

(2) Now let X be the increasing, continuous, $\mathbf{F}$-adapted process, for which we want to prove the assertion of the theorem. In this step of the proof we also assume in addition that X is square integrable. For every $m \geq 1$ we define the process $X^m = (X^m_t)_{t\geq 0}$ by

$$X^m := \sum_{j\geq 1} X_{t^m_{j-1}}\,1_{[t^m_{j-1},t^m_j[}\,.$$

To prove the asserted $\sigma(L^1, L^\infty)$-convergence, let Z be an arbitrary given element of L^∞. Then

$$\mathbf{E}\{Z\,C^n(X)_t\} = \mathbf{E}\{Z\,C^n(X^m)_t\} + R_1^{m,n}(t)\,, \tag{3.4.4}$$

where

$$R_1^{m,n}(t) = \mathbf{E}\{Z\,(C^n(X)_t - C^n(X^m)_t)\}\,.$$

First we compute $\mathbf{E}\{Z\,C^n(X^m)_t\}$:

$$\begin{aligned}&\mathbf{E}\{Z\,C^n(X^m)_t\}\\ &\quad= \mathbf{E}\big\{Z\sum_{k=1}^{m(t)}\mathbf{E}\{X_{t^n_k} - X_{t^n_{k-1}}\,|\,\mathcal{H}^n_{t^n_{k-1}}\}\big\}\\ &\quad= \mathbf{E}\big\{\sum_{k=1}^{m(t)}\mathbf{E}\{Z\,|\,\mathcal{H}^n_{t^n_{k-1}}\}(X_{t^n_k} - X_{t^n_{k-1}})\big\}\,,\end{aligned}$$

and with the definition

$$F_n \;:=\; \sum_{k=1}^{m(t)} \mathbf{E}\{Z \mid \mathcal{H}^n_{t^n_{k-1}}\}\, 1_{[t^m_{k-1}, t^m_k[}$$

we have an $\mathbf{F}$-predictable function such that

$$\mathbf{E}\{Z\, C^n(X^m)_t\} \;=\; \mathbf{E}\Big\{ \int_0^t F_n(s)\, dX^m_s \Big\} . \tag{3.4.5}$$

The increasing process X^m corresponds to the measure

$$\sum_{j\geq 1} (X_{t^m_j} - X_{t^m_{j-1}})\, \delta_{t^m_j} .$$

Hence, if $k(j)$ denotes that integer for which

$$t^n_{k(j)} \;=\; t^m_j \quad (n \geq m) ,$$

then

$$\begin{aligned} &\int_0^t F_n(s)\, dX^m_s \\ &\quad = \sum_{j\geq 1} F_n(t^n_{k(j)})(X_{t^m_j} - X_{t^m_{j-1}}) \\ &\quad = \sum_{j\geq 1} 1_{[0,t]}(t^m_j) \mathbf{E}\{Z|\mathcal{H}^n_{t^n_{k(j)-1}}\}(X_{t^m_j} - X_{t^m_{j-1}}) . \end{aligned}$$

Thus we have

$$\mathbf{E}\{Z\, C^n(X^m)_t\} = \mathbf{E}\Big(\sum_{j\geq 1} 1_{[0,t]}(t^m_j) \mathbf{E}\{Z|\mathcal{F}_{t^m_j -}\}(X_{t^m_j} - X_{t^m_{j-1}}) \Big) \;+\; R^{m,n}_2(t) , \tag{3.4.6}$$

with

$$R^{m,n}_2(t) := \mathbf{E}\Big(\sum_{j\geq 1} 1_{[0,t]}(t^m_j) \big[\mathbf{E}\{Z|\mathcal{H}^n_{t^n_{k(j)-1}}\} \;-\; \mathbf{E}\{Z|\mathcal{F}_{t^m_j -}\} \big]\, (X_{t^m_j} - X_{t^m_{j-1}}) \Big) .$$

The continuity of X implies that every X_t is $\mathcal{F}_{t-}$-measurable. This gives

$$\mathbf{E}\big(\mathbf{E}\{Z|\mathcal{F}_{t^m_j -}\}(X_{t^m_j} - X_{t^m_{j-1}}) \big) \;=\; \mathbf{E}\big\{ Z\, (X_{t^m_j} - X_{t^m_{j-1}}) \big\} ,$$

and we get from (3.4.6) that

$$\mathbf{E}\{Z\, C^n(X^m)_t\} \;=\; \mathbf{E}\{Z\, (X_t - X_0)\} \;+\; R^{m,n}_2(t)$$

and altogether we have proved that

$$\mathbf{E}\{Z\, C^n(X)_t\} = \mathbf{E}\{Z\, (X_t - X_0)\} \;+\; R^{m,n}_1(t) \;+\; R^{m,n}_2(t) \tag{3.4.7}$$

for $m \le n$. So we have to prove for the asserted $\sigma(L^1, L^\infty)$-convergence that for every $\varepsilon > 0$ there exists an $m \ge 1$ such that for every $n \ge m$ we have

$$|R_1^{m,n}(t)| < \varepsilon \quad \text{and} \quad |R_2^{m,n}(t)| < \varepsilon \,.$$

For $R_1^{m,n}(t)$ we use the inequality (3.4.3). With $D := \|Z\|_\infty$ we get

$$\begin{aligned} |R_1^{m,n}(t)| &\le D\,\mathbf{E}|C^n(X)_t - C^n(X^m)_t| \\ &= D\,\mathbf{E}\,C^n(X - X^m)_t \\ &\le D\left(\mathbf{E}\{C^n(X - X^m)_t\}^2\right)^{\frac{1}{2}} \end{aligned}$$

and hence

$$\begin{aligned} &R_1^{m,n}(t)^4 \\ &\le D^4\left(\mathbf{E}\{C^n(X - X^m)_t\}^2\right)^2 \\ &\le 2^4 D^4 \mathbf{E}\Big\{\max_{0\le k<m(t)} \big((X - X^m)_{t^n_{m(t)}} - (X - X^m)_{t^n_k}\big)^2\Big\} \\ &\qquad\qquad \cdot \big([\mathbf{E}(X_t - X_0)^2]^{\frac{1}{2}} + [\mathbf{E}(X_t^m - X_0^m)^2]^{\frac{1}{2}}\big)^2 \\ &\le 2^6 D^4 \mathbf{E}(X_t - X_0)^2\,\mathbf{E}\Big\{\max_{0\le k<m(t)} \big((X - X^m)_{t^n_{m(t)}} - (X - X^m)_{t^n_k}\big)^2\Big\} \\ &\le 2^7 D^4 \mathbf{E}(X_t - X_0)^2\,\mathbf{E}\Big\{\max_{0\le k\le m(t)} \big((X - X^m)_{t^n_k}\big)^2\Big\} \\ &\le 2^7 D^4 \mathbf{E}(X_t - X_0)^2\,\mathbf{E}\Big\{\max_{j\ge 1} 1_{[0,t]}(t_j^m)(X_{t_j^m} - X_{t_{j-1}^m})^2\Big\}\,. \end{aligned}$$

Since X is assumed to be continuous, it follows easily that

$$\lim_{m\to\infty} \mathbf{E}\Big\{\max_{j\ge 1} 1_{[0,t]}(t_j^m)(X_{t_j^m} - X_{t_{j-1}^m})^2\Big\} \;=\; 0\,,$$

and thus we have proved that there exists an $m_1 = m_1(\varepsilon)$ such that for all $n \ge m \ge m_1$ we have $|R_1^{m,n}(t)| < \varepsilon$.

For $R_2^{m,n}(t)$ we get from the definition

$$R_2^{m,n}(t)^2 \le \mathbf{E}(X_t - X_0)^2\,\mathbf{E}\Big\{\max_{j\ge 1}\Big(1_{[0,t]}(t_j^m)\big(\mathbf{E}\{Z\,|\mathcal{H}^n_{t^n_{k(j)-1}}\} \;-\; \mathbf{E}\{Z|\mathcal{F}_{t_j^m-}\}\big)\Big)^2\Big\}\,.$$

Since $t^n_{k(j)-1} \nearrow t_j^m$ for $n \to \infty$, we have

$$\bigvee_{n\ge 1} \mathcal{H}^n_{t^n_{k(j)-1}} \;=\; \mathcal{H}^\infty_{t_j^m-}\,,$$

and it is easy to show that $\mathbf{E}\{Z|\mathcal{H}^\infty_{t-}\} = \mathbf{E}\{Z|\mathcal{F}_{t-}\}$ for all $t > 0$. Hence

$$\lim_{n\to\infty} \mathbf{E}\{Z|\mathcal{H}^n_{t^n_{k(j)-1}}\} \;=\; \mathbf{E}\{Z|\mathcal{F}_{t_j^m-}\}$$

P-a.s. for all $j \geq 1$. Since $Z \in L^\infty$, it follows from Lebesgue's theorem that $\lim_{n\to\infty} R_2^{m,n}(t) = 0$. Hence for a given $\varepsilon > 0$ and $m \geq 1$ there exists an $n(\varepsilon, m) \geq m$ such that

$$|R_2^{m,n}(t)| \;<\; \varepsilon$$

for all $n \geq n(\varepsilon, m)$.

So let $\varepsilon > 0$ be given. Then first we can choose an $m(\varepsilon)$ such that $|R_1^{m(\varepsilon),n}(t)| < \varepsilon$ for all $n \geq m(\varepsilon)$. Then we choose an $n(\varepsilon) := n(\varepsilon, m(\varepsilon)) \geq m(\varepsilon)$ such that $|R_2^{m(\varepsilon),n}(t)| \;<\; \varepsilon$ for all $n \geq n(\varepsilon)$. Then we obtain from (3.4.7)

$$|\,\mathbf{E}\{Z\, C^n(X)_t\} \;-\; \mathbf{E}\{Z\,(X_t - X_0)\}\,| \;<\; 2\varepsilon$$

for all $n \geq n(\varepsilon)$. Since $\varepsilon > 0$ and $Z \in L^\infty$ were arbitrarily chosen, we have proved the asserted $\sigma(L^1, L^\infty)$-convergence under the assumption that X is square integrable.

(3) Now just suppose that X is integrable. For $Z \in L^\infty$ and $\varepsilon > 0$ given we choose a constant $c > 0$ such that

$$\mathbf{E}\{X_t \;-\; (X \wedge c)_t\} \;<\; \frac{\varepsilon}{2\|Z\|_\infty}\,.$$

Then we have

$$\mathbf{E}\{Z\, C^n(X)_t\} - \mathbf{E}\{Z\,(X_t - X_0)\} = \mathbf{E}\{Z\, C^n(X \wedge c)_t\} - \mathbf{E}\{Z((X \wedge c)_t - (X \wedge c)_0)\} + R_t^{n,c}\,,$$

where

$$R_t^{n,c} \;=\; \mathbf{E}\{Z(C^n(X)_t \;-\; C^n(X \wedge c)_t)\} \;+\; \mathbf{E}\{Z\,((X_t - (X \wedge c)_t) - (X_0 - (X \wedge c)_0))\}\,.$$

Then

$$|R_t^{n,c}| \;\leq 2\|Z\|_\infty \mathbf{E}(X \;-\; (X \wedge c))_t \;<\; \varepsilon\,.$$

This proves finally the assertion of the theorem for the general case that X is only assumed to be integrable. □

One special application of the theorem is formulated in the following corollary.

Corollary 3.5 *Let $M = (M_t)_{t\geq 0}$ be a square integrable martingale and let $\mathbf{F}$ be the standard filtration generated by the canonical filtration of M. Let as before $(\pi_n)_{n\geq 1}$ denote a null-sequence of partitions and define for every $n \geq 1$ the filtration $\mathbf{H}^n = (\mathcal{H}_t^n)_{t\geq 0}$ by*

$$\mathcal{H}_t^n \;:=\; \sigma\big(\{M_{t_k^n} \,|\, t_k^n \leq t\}\big)\;.$$

If the predictable quadratic variation $<\!M\!>$ of M is continuous, then

$$\lim_{n\to\infty} \sum_{k\geq 1} 1_{[0,t]}(t_k^n)\mathbf{E}\big\{(M_{t_k^n} - M_{t_{k-1}^n})^2 \,|\, \mathcal{H}_{t_{k-1}^n}^n\big\} \;= <\!M\!>_t$$

for every $t > 0$ in the topology $\sigma(L^1, L^\infty)$.

To get convergence in the L^1-norm, it seems that in general one has to assume a stronger condition on the limit process X than just continuity. In principle, it will be a condition implying that there is a kind of uniform nearness of the conditional expectations $\mathbf{E}\{X_{t_k^n} - X_{t_{k-1}^n} | \mathcal{H}^n_{t_{k-1}^n}\}$ to $\mathbf{E}\{X_{t_k^n} - X_{t_{k-1}^n} | \mathcal{F}_{t_{k-1}^n}\}$. The condition which we impose on X in the following theorem is a condition which is usually fulfilled in applications (survival analysis, queuing theory, risk theory etc.).

Theorem 3.6 *Let $(\mathbf{H}^n)_{n\geq 1}$ and $\mathbf{F}$ be given as before, and let X be an increasing, right continuous, $\mathbf{F}$-adapted and integrable process. Suppose that in addition X has $\mathbf{P}$-a.s. a density Z relative to a non-random measure μ on $\mathbf{R}_+$, i.e.*

$$X_t \;=\; \int_0^t Z_s\,\mu(ds)$$

for every $t > 0$ $\mathbf{P}$-a.s.. Then $\lim_{n\to\infty} C^n(X)_t \;=\; X_t$ in the L^1-norm for every $t > 0$.

Proof: (1) We will denote by $\mathcal{P}$ the predictable σ-algebra on $\mathbf{R}_+ \times \Omega$ for the filtration $\mathbf{F}$. For every $n \geq 1$ we define then $\mathcal{P}_n$ as the sub-σ-algebra of $\mathcal{P}$ generated by the sets $\{0\} \times A$, where $A \in \mathcal{H}^n_0$ and by the sets $]t^n_{k-1}, t^n_k] \times A$, where $A \in \mathcal{H}^n_{t^n_{k-1}}$ and $k \geq 1$. Furthermore, we define a sequence $(M_n)_{n\geq 1}$ of maps $M_n : \mathbf{R}_+ \times \Omega \;\to\; \mathbf{R}_+$ by

$$M_n \;:=: \sum_{k\geq 1} \frac{\mathbf{E}\{X_{t_k^n} - X_{t_{k-1}^n} | \mathcal{H}^n_{t_{k-1}^n}\}}{\mu(]t^n_{k-1}, t^n_k])}\, 1_{]t^n_{k-1}, t^n_k]}\,, \tag{3.6.1}$$

with the (admissible) convention that

$$\frac{\mathbf{E}\{X_{t_k^n} - X_{t_{k-1}^n} | \mathcal{H}^n_{t_{k-1}^n}\}}{\mu(]t^n_{k-1}, t^n_k])} \;:=\; 0\;\;,$$

in case that $\mu(]t^n_{k-1}, t^n_k]) \;=\; 0$. Now let $t > 0$ be given. We set $\mathcal{P}^t := \mathcal{P} \cap ([0,t] \times \Omega)$ and $\mathcal{P}^t_n := \mathcal{P}_n \cap ([0,t] \times \Omega)$ for $n \geq 1$. Then we define the probability space

$$\Omega^t \;:=\; \Big([0,t] \times \Omega, \mathcal{P}^t, \frac{\mu}{\mu([0,t])} \otimes \mathbf{P}\Big)\,.$$

In the following we will view $(M_n)_{n\geq 1}$ as a process on Ω^t. By definition, every M_n is $\mathcal{P}^t_n$-measurable, i.e. $(M_n)_{n\geq 1}$ is a $(\mathcal{P}^t_n)$-adapted process. It is also clear that $(M_n)_{n\geq 1}$ is integrable. We will prove now that $(M_n)_{n\geq 1}$ is even a $(\mathcal{P}^t_n)$-martingale.

We set $\mu_t \;:=\; \dfrac{\mu}{\mu([0,t])}$. For $m < n$ we take a set $A \in \mathcal{H}^m_{t^m_{j-1}}$. Furthermore, we set $I^m_j := \{k \geq 1 |\, t^m_{j-1} < t^n_k \leq t^m_j\}$ and

$$Y^n_k \;:=\; \frac{\mathbf{E}\{X_{t_k^n} - X_{t_{k-1}^n} | \mathcal{H}^n_{t_{k-1}^n}\}}{\mu(]t^n_{k-1}, t^n_k])}\,.$$

Then

$$
\begin{aligned}
&\int_{]t^m_{j-1},t^m_j]\times A} M_n\, d(\mu_t\otimes\mathbf{P})\\
&= \int 1_{]t^m_{j-1},t^m_j]\times A}\big(\sum_{k\in I^m_j} Y^n_k\, 1_{]t^n_{k-1},t^n_k]}\big)\, d(\mu_t\otimes\mathbf{P})\\
&= \sum_{k\in I^m_j}\int_\Omega\int_{\mathbf{R}_+} 1_A\, Y^n_k\, 1_{]t^n_{k-1},t^n_k]}\, d\mu_t\, d\mathbf{P}\\
&= \sum_{k\in I^m_j}\mu_t(]t^n_{k-1},t^n_k])\int_A Y^n_k\, d\mathbf{P}\\
&= \sum_{k\in I^m_j}\frac{\mu_t(]t^n_{k-1},t^n_k])}{\mu(]t^n_{k-1},t^n_k])}\int_A \mathbf{E}\{X_{t^n_k}-X_{t^n_{k-1}}|\mathcal{H}^n_{t^n_{k-1}}\}\, d\mathbf{P}\\
&= \mu([0,t])^{-1}\int_A \mathbf{E}\{X_{t^m_j}-X_{t^m_{j-1}}|\mathcal{H}^m_{t^m_{j-1}}\}\, d\mathbf{P}\\
&= \frac{\mu(]t^m_{j-1},t^m_j])}{\mu([0,t])}\int_A \frac{\mathbf{E}\{X_{t^m_j}-X_{t^m_{j-1}}|\mathcal{H}^m_{t^m_{j-1}}\}}{\mu(]t^m_{j-1},t^m_j])}\, d\mathbf{P}\\
&= \int_\Omega\int_{\mathbf{R}_+} 1_A\, 1_{]t^m_{j-1},t^m_j]}\, d\mu_t\, d\mathbf{P}\\
&= \int_{]t^m_{j-1},t^m_j]\times A} M_m\, d(\mu_t\otimes\mathbf{P})\,.
\end{aligned}
$$

Since the sets $]t^m_{j-1},t^m_j]\times A$ ($j\geq 1$, $A\in\mathcal{H}^m_{t^m_{j-1}}$) together with the sets $\{0\}\times A$ ($A\in\mathcal{H}^m_0$) form a $\cap$-stable system generating $\mathcal{P}_m$, it follows from the just proved equation that $(M_n)_{n\geq 1}$ is a $(\mathcal{P}^t_n)$-martingale.

Since in addition, $(M_n)_{n\geq 1}$ is non-negative, the martingale convergence theorem implies that $(M_n)_{n\geq 1}$ is a.s. convergent.

(2) Now we prove that $(M_n)_{n\geq 1}$ also converges in $L^1([0,t]\times\Omega)$. By (1) it is necessary and sufficient to show that $(M_n)_{n\geq 1}$ is uniformly integrable. We prove this under the momentary extra condition that the process $(Z_s)_{0\leq s\leq t}$ belongs to $L^2([0,t]\times\Omega)$. Then it is sufficient to prove that

$$\sup_{n\geq 1}\int M_n^2\, d(\mu_t\otimes\mathbf{P})\ <\ \infty\,.$$

For every $s\in]t^n_{k-1},t^n_k]$ we have

$$
\begin{aligned}
M_n(s,\cdot) &= \frac{\mathbf{E}\{X_{t^n_k}-X_{t^n_{k-1}}|\mathcal{H}^n_{t^n_{k-1}}\}}{\mu(]t^n_{k-1},t^n_k])}\\
&= \frac{\int_{]t^n_{k-1},t^n_k]}\mathbf{E}\{Z_r\,|\mathcal{H}^n_{t^n_{k-1}}\}\,\mu(dr)}{\mu(]t^n_{k-1},t^n_k])}\,,
\end{aligned}
$$

and hence

$$M_n(s,\cdot)^2 \;\le\; \frac{\int_{]t_{k-1}^n,t_k^n]} \big(\mathbf{E}\{Z_r\,|\mathcal{H}^n_{t^n_{k-1}}\}\big)^2\,\mu(dr)}{\mu(]t^n_{k-1},t^n_k])} \;\le\; \frac{\int_{]t_{k-1}^n,t_k^n]} \mathbf{E}\{Z_r^2\,|\mathcal{H}^n_{t^n_{k-1}}\}\,\mu(dr)}{\mu(]t^n_{k-1},t^n_k])}\,.$$

Therefore we obtain

$$\begin{aligned}
&\int M_n^2\,d(\mu_t\otimes\mathbf{P})\\
&\le \int_{[0,t]}\Big(\sum_{k\ge1}\mu(]t^n_{k-1},t^n_k])^{-1}1_{]t^n_{k-1},t^n_k]}(u)\int_{]t^n_{k-1},t^n_k]}\mathbf{E}\big(\mathbf{E}\{Z_r^2\,|\mathcal{H}^n_{t^n_{k-1}}\}\big)\,\mu(dr)\Big)\,\mu_t(du)\\
&= \int_{[0,t]}\Big(\sum_{k\ge1}\mu(]t^n_{k-1},t^n_k])^{-1}1_{]t^n_{k-1},t^n_k]}(u)\big(\int_{]t^n_{k-1},t^n_k]}\mathbf{E}\,Z_r^2\,\mu(dr)\big)\Big)\,\mu_t(du)\\
&= \int_{[0,t]} Z^2\,d(\mu_t\otimes\mathbf{P})\,,
\end{aligned}$$

and it follows that $(M_n)_{n\ge1}$ is uniformly integrable. Since $(M_n)_{n\ge1}$ is also $(\mu_t\otimes\mathbf{P})$-a.s. convergent, there exists an $L^1([0,t]\times\Omega)$-limit M of $(M_n)_{n\ge1}$. Because of

$$\mathbf{E}\,\Big|\int_{[0,t]} M_n\,d\mu_t \;-\; \int_{[0,t]} M\,d\mu_t\Big| \;\le\; \mathbf{E}\int_{[0,t]}|M_n-M|\,d\mu_t$$

and the observation that

$$\begin{aligned}
&\int_{[0,t]} M_n\,d\mu_t\\
&\quad= \mu([0,t])^{-1}\sum_{k\ge1}1_{[0,t]}(t^n_k)\mathbf{E}\{X_{t^n_k}-X_{t^n_{k-1}}|\mathcal{H}^n_{t^n_{k-1}}\}\\
&\quad= \mu([0,t])^{-1}\,C^n(X)_t\,,
\end{aligned}$$

we have especially proved that

$$\lim_{n\to\infty} C^n(X)_t \;=\; X_t$$

in the L^1-norm for every $t>0$.

(3) Finally, if $(Z_s)_{0\le s\le t}$ is not assumed to be square integrable, we set for every constant $c>0$

$$X_t^c \;:=\; \int_{[0,t]}(Z_s\wedge c)\,\mu(ds)\,.$$

For any given $\varepsilon>0$ it is then possible to find a $c=c(\varepsilon)$ such that

$$\mathbf{E}\,X_t \;-\; \mathbf{E}\,X_t^c \;<\; \varepsilon\,.$$

If we denote by M_n^c the map of step (2) associated with X^c, then $(M_n^c)_{n\geq 1}$ converges in $L^1([0,t]\times\Omega)$ by step (2). Furthermore, we have

$$\begin{aligned}
&\int_{[0,t]\times\Omega} (M_n \,-\, M_n^c)\, d(\mu_t\otimes\mathbf{P}) \\
&\leq\ \int_{[0,t]} \Bigg(\sum_{k\geq 1} \mu(]t_{k-1}^n, t_k^n])^{-1} 1_{]t_{k-1}^n, t_k^n]}(u) \\
&\qquad\qquad \cdot \int_\Omega \int_{]t_{k-1}^n, t_k^n]} \left| \mathbf{E}\{Z_r | \mathcal{H}^n_{t^n_{k-1}}\} \,-\, \mathbf{E}\{Z_r\wedge c | \mathcal{H}^n_{t^n_{k-1}}\}\right| \mu(dr)\, d\mathbf{P}\Bigg)\, \mu_t(du) \\
&\leq\ \int_\Omega \int_{[0,t]} (Z_r \,-\, Z_r\wedge c)\, \mu_t(dr)\, d\mathbf{P} \\
&=\ \mu([0,t])^{-1}\mathbf{E}(X_t \,-\, X_t^c)\,.
\end{aligned}$$

Now it follows easily that $(M_n)_{n\geq 1}$ is a Cauchy sequence in $L^1([0,t]\times\Omega)$ and hence convergent. As in the last part of step (2) one finally gets $\lim_{n\to\infty} C^n(X)_t \,=\, X_t$ in $L^1(\Omega)$. This finishes the proof of the theorem. □

4 Existence of the Second Order Characteristics

In this section we will prove that in the sense of theorem 2.2 all second order characteristics exist for every L^4-martingale. So let $M \,=\, (M_t)_{t\geq 0}$ be in the following a fixed cadlag $\mathbf{F}$-martingale such that $M_t \in L^4(\Omega)$ for all $t \geq 0$. We will often make use of the following structure of M (cf. [3], ch.4). First we have the decomposition

$$M \,=\, M^c \,+\, M^d$$

into two L^4-martingales. M^c is a continuous martingale, called the *continuous part* of M. The martingale M^d, called the *purely discontinuous part* of M, has the structure

$$M^d \,=\, M^p \,+\, M^t\,,$$

where the martingales M^p and M^t are called the *predictable* and the *totally inaccessible part* resp. of M. We have

$$M^p \,=\, \sum_{i\geq 1} X_i\, 1_{[S_i,\infty[}\,,$$

where the S_i are predictable stopping times and $X_i \,=\, \triangle M_{S_i}$. M^t has the structure

$$M^t \,=\, \sum_{j\geq 1} \left(Y_j\, 1_{[T_j,\infty[} \,-\, \tilde{A}^j\right).$$

Here the T_j are totally inaccessible stopping times, $\tilde{A}^j$ denotes the (continuous) compensator of $Y_j\, 1_{[T_j,\infty[}$ and $Y_j \,=\, \triangle M_{T_j}$. Moreover, the family

$$\{\, T \,|\, T = S_i \text{ or } T = T_j,\ i,j\in\mathbf{N}\,\}$$

may and will be assumed to have pairwise disjoint graphs. Then the following orthogonality relation holds:

$$\mathbf{E}\, M_t^2 \;=\; \mathbf{E}\,(M_t^c)^2 + \sum_{i\geq 1} \mathbf{E}\big(X_i^2\, 1_{[S_i\leq t]}\big) + \sum_{j\geq 1} \mathbf{E}\big(Y_j\, 1_{[T_j\leq t]} \;-\; \tilde{A}_t^j\big)^2 .$$

Theorem 4.1 (Existence of $(M)^{(4)}$) *For every $t \geq 0$,*

$$(M)_t^{(4)} \;:=\; \lim_{n\to\infty} \sum_{k\geq 1} 1_{[0,t]}(t_k^n)\,(M_{t_k^n} \;-\; M_{t_{k-1}^n})^4$$

exists in L^1 and (with the notations introduced above)

$$(M)_t^{(4)} \;=\; \sum_{i\geq 1} X_i^4\, 1_{[S_i\leq t]} \;+\; \sum_{j\geq 1} Y_j^4\, 1_{[T_j\leq t]} .$$

Proof: (1) First we assume that M^c and M^d are bounded by some constant $C > 0$. For every $n \geq 1$ we set as before

$$m(t) = m_n(t) := \max\{k \geq 1 | t_k^n \leq t\} .$$

Then we have

$$\sum_{k=1}^{m(t)} (M_{t_k^n} - M_{t_{k-1}^n})^4 \;= \sum_{k=1}^{m(t)} (M_{t_k^n}^d - M_{t_{k-1}^n}^d)^4 \;+\; \sum_{i=1}^{4} A_i^n(t) ,$$

where

$$\begin{aligned}
A_1^n(t) \;&:=\; 4\sum_{k=1}^{m(t)} (M_{t_k^n}^c - M_{t_{k-1}^n}^c)(M_{t_k^n}^d - M_{t_{k-1}^n}^d)^3 ,\\
A_2^n(t) \;&:=\; 6\sum_{k=1}^{m(t)} (M_{t_k^n}^c - M_{t_{k-1}^n}^c)^2(M_{t_k^n}^d - M_{t_{k-1}^n}^d)^2 ,\\
A_3^n(t) \;&:=\; 4\sum_{k=1}^{m(t)} (M_{t_k^n}^c - M_{t_{k-1}^n}^c)^3(M_{t_k^n}^d - M_{t_{k-1}^n}^d) , \ \text{ and}\\
A_4^n(t) \;&:=\; \sum_{k=1}^{m(t)} (M_{t_k^n}^c - M_{t_{k-1}^n}^c)^4 .
\end{aligned}$$

We assert that $\lim_{n\to\infty} A_i^n(t) = 0$ in $L^1(\Omega)$ for $i = 1, 2, 3, 4$. Since the proofs are very similar, we only show that $\lim_{n\to\infty} A_1^n(t) = 0$ in $L^1(\Omega)$. For the proof of that assertion we will use the obvious inequality

$$|A_1^n(t)| \leq 8C \max_k |M_{t_k^n}^c - M_{t_{k-1}^n}^c| \sum_{k=1}^{m(t)} (M_{t_k^n}^d - M_{t_{k-1}^n}^d)^2 .$$

Now let $\varepsilon > 0$ be given. Since

$$\lim_{n\to\infty} \sum_{k=1}^{m(t)} (M^d_{t^n_k} - M^d_{t^n_{k-1}})^2 \;=\; [M^d]_t$$

in $L^1(\Omega)$, we can find an $n_1 \in \mathbf{N}$ such that

$$\mathbf{E}\Big| \sum_{k=1}^{m(t)} (M^d_{t^n_k} - M^d_{t^n_{k-1}})^2 \;-\; [M^d]_t \Big| \;<\; \frac{\varepsilon}{48\, C^2}$$

for all $n \geq n_1$. For every $m \geq 1$ and $\eta > 0$ we set

$$\Omega_{m,\eta} := \Big\{ \omega \mid \max_k |M^c_{t^n_k}(\omega) - M^c_{t^n_{k-1}}(\omega)| \;<\; \eta \text{ for all } n \geq m \Big\}.$$

Then the sequence $(\Omega_{m,\eta})_{m\geq 1}$ is increasing with $\bigcup_{m\geq 1} \Omega_{m,\eta} \;=\; \Omega$. We choose

$$\eta \;=\; \frac{1}{24\, C\, \mathbf{E}\, [M^d]_t} \;\;.$$

Then there exists an $m \in \mathbf{N}$ such that

$$\mathbf{E}\big\{ 1_{\Omega^c_{m,\eta}} [M^d]_t \big\} \;<\; \frac{\varepsilon}{48\, C^2}\,.$$

With $n_0 := \max(n_1, m)$ we have for all $n \geq n_0$

$$\begin{aligned}
&\mathbf{E}\, |A^n_1(t)| \\
&\leq\;\; 16\, C^2\, \mathbf{E}\Big| \sum_{k=1}^{m(t)} (M^d_{t^n_k} - M^d_{t^n_{k-1}})^2 - [M^d]_t \Big| + 8\, C\, \mathbf{E}\big(\max_k |M^c_{t^n_k} - M^c_{t^n_{k-1}}| \cdot [M^d]_t \big) \\
&<\;\; \frac{\varepsilon}{3} \;+\; 8\, C \int_{\Omega_{m,\eta}} \max_k |M^c_{t^n_k} - M^c_{t^n_{k-1}}| \cdot [M^d]_t \, d\mathbf{P} \;+\; 16\, C^2 \int_{\Omega^c_{m,\eta}} [M^d]_t \, d\mathbf{P} \\
&<\;\; \frac{\varepsilon}{3} \;+\; 8\, C\, \eta\, \mathbf{E}[M^d]_t \;+\; 16\, C^2\, \mathbf{E}\big\{ 1_{\Omega^c_{m,\eta}} [M^d]_t \big\} \;\;<\; \varepsilon
\end{aligned}$$

by the choice of n_0. Since $\varepsilon > 0$ was arbitrary, we have proved that $\lim_{n\to\infty} A^n_1(t) \;=\; 0$ in $L^1(\Omega)$ and $\lim_{n\to\infty} A^n_i(t) \;=\; 0$ $(i = 2, 3, 4)$ follows similarly.

It remains to prove the existence of

$$\lim_{n\to\infty} \sum_{k=1}^{m(t)} (M^d_{t^n_k} - M^d_{t^n_{k-1}})^4$$

in $L^1(\Omega)$. This is proved similarly to the proof of the existence of $[M^d]$ in [3] (th. 18.6). So let us assume first that the number of jumps of M on $[0, t]$ is bounded, which means that

$$M^d_s \;=\; \sum_{i=1}^{k_1} X_i\, 1_{[S_i \leq s]} \;+\; \sum_{j=1}^{k_2} (Y_j\, 1_{[T_i \leq s]} - \tilde{A}^j_s)$$

for $k_1, k_2 \in \mathbf{N}$ and $s \le t$. Since $(M_s^d)_{s\le t}$ is now a process of finite variation having only finite many jumps, one can prove that even pathwise

$$\lim_{n\to\infty} \sum_{k=1}^{m(t)} (M^d_{t^n_k} - M^d_{t^n_{k-1}})^4 \;=\; \sum_{i=1}^{k_1} X_i^4 , 1_{[S_i \le t]} \;+\; \sum_{j=1}^{k_2} Y_j^4\, 1_{[T_i \le t]}\,,$$

and the boundedness assumption on M implies that this convergence also takes place in the L^1-norm.

If the number of jumps of M on $[0,t]$ is not bounded, we set

$$M^{d,m} \;:=\; \sum_{i=1}^{m} Z_i\,,$$

where every Z_i is either of the form $X_i 1_{[S_i,\infty[}$ or of the form $(Y_j 1_{[T_j,\infty[} - \tilde{A}^j)$. For a shorter notation we will write

$$(N)_t^{(4),n} \;:=\; \sum_{k=0}^{m(t)} (N_{t^n_k} - N_{t^n_{k-1}})^4$$

for every L^4-martingale $N = (N_t)_{t\ge 0}$. Now set

$$\bar{M}^{d,m} \;:=\; M^d - M^{d,m}\,.$$

Then

$$(M^d)_t^{(4),n} \;-\; (M^{d,m})_t^{(4),n} \;=\; \sum_{i=1}^{4} B_i^{n,m}\,,$$

where

$$\begin{aligned}
B_1^{n,m}(t) \;&:=\; 4\sum_{k=1}^{m(t)} (M^{d,m}_{t^n_k} - M^{d,m}_{t^n_{k-1}})^3 (\bar{M}^{d,m}_{t^n_k} - \bar{M}^{d,m}_{t^n_{k-1}})\,,\\
B_2^{n,m}(t) \;&:=\; 6\sum_{k=1}^{m(t)} (M^{d,m}_{t^n_k} - M^{d,m}_{t^n_{k-1}})^2 (\bar{M}^{d,m}_{t^n_k} - \bar{M}^{d,m}_{t^n_{k-1}})^2\,,\\
B_3^{n,m}(t) \;&:=\; 4\sum_{k=1}^{m(t)} (M^{d,m}_{t^n_k} - M^{d,m}_{t^n_{k-1}}) (\bar{M}^{d,m}_{t^n_k} - \bar{M}^{d,m}_{t^n_{k-1}})^3\,, \text{ and}\\
B_4^{n,m}(t) \;&:=\; \sum_{k=1}^{m(t)} (\bar{M}^{d,m}_{t^n_k} - \bar{M}^{d,m}_{t^n_{k-1}})^4\,,
\end{aligned}$$

and we prove that $\lim_{m\to\infty} B_i^{n,m} \;=\; 0$ in L^1 for $i = 1,2,3,4$ uniformly in n. By the boundedness assumption we have

$$\begin{aligned}
|B_1^{n,m}| \;&\le\; 4\,C\big(\sum_{k=1}^{m(t)} (M^{d,m}_{t^n_k} - M^{d,m}_{t^n_{k-1}})^2\big) \max_k |\bar{M}^{d,m}_{t^n_k} - \bar{M}^{d,m}_{t^n_{k-1}}|\\
&\le\; 8\,C \sup_{s\le t} |\bar{M}^{d,m}_s| \sum_{k=1}^{m(t)} (M^{d,m}_{t^n_k} - M^{d,m}_{t^n_{k-1}})^2\,.
\end{aligned}$$

It follows from the Burkholder-Davis-Gundy-inequality (applied to $M^{d,m}$) that

$$\mathbf{E}|B_1^{n,m}| \;\leq\; K\left(\mathbf{E}\sup_{s\leq t}(M_s^{d,m})^4\right)^{\frac{1}{2}}\left(\mathbf{E}\sup_{s\leq t}(\bar{M}_s^{d,m})^2\right)^{\frac{1}{2}}$$

for some constant $K > 0$. Since

$$\lim_{m\to\infty}\mathbf{E}\sup_{s\leq t}(\bar{M}_s^{d,m})^2 \;=\; 0\,,$$

we have proved that $\lim_{m\to\infty} B_1^{n,m} \;=\; 0$ in L^1 uniformly in n. Similar proofs show that also $\lim_{m\to\infty} B_i^{n,m} \;=\; 0$ $(i = 2,3,4)$ in L^1 uniformly in n. Hence under the boundedness assumption on M we have proved altogether that the sequence $((M)_t^{(4),n})_{n\geq 1}$ converges in L^1 and has the asserted limit.

(2) If M^c and M^d are not necessarily bounded, we introduce for every $C > 0$ the set

$$F_C \;:=\; \left\{\,\omega \mid \sup_{s\leq t}|M_s^c(\omega) \leq C \;\text{ and }\; \sup_{s\leq t}|M_s^d(\omega) \leq C\,\right\}.$$

Similar as in (1) one can prove that

$$\lim_{n\to\infty} 1_{F_C}\,(M)_t^{(4),n}$$

exists in L^1. Since $\lim_{C\nearrow\infty}\mathbf{P}\,F_C^c \;=\; 0$, it follows that the sequence $((M)_t^{(4),n})_{n\geq 1}$ converges in probability. For the asserted L^1-convergence it is therefore sufficient to show that $((M)_t^{(4),n})_{n\geq 1}$ is uniformly integrable. Now, for every $A \in \mathcal{F}$ we have

$$\begin{aligned}
&\mathbf{E}\big(1_A\sum_{k=1}^{m(t)}(M_{t_k^n} - M_{t_{k-1}^n})^4\big)\\
&\leq\; 2\,\mathbf{E}\big(1_A\sup_{s\leq t}M_s^2\sum_{k=1}^{m(t)}(M_{t_k^n} - M_{t_{k-1}^n})^2\big)\\
&\leq\; 2\,K\left(\mathbf{E}\,1_A\sup_{s\leq t}M_s^4\right)^{\frac{1}{2}}\left(\mathbf{E}\sup_{s\leq t}M_s^4\right)^{\frac{1}{2}},
\end{aligned}$$

where the last line follows again from the Burkholder-Davis-Gundy-inequality. The uniform integrability is now obvious and the theorem is proved. □

Theorem 4.2 (Existence of $(M)^{(\tilde{4})}$) *For every $t \geq 0$,*

$$(M)_t^{(\tilde{4})} \;:=\; \lim_{n\to\infty}\sum_{k=1}^{m(t)}\mathbf{E}\big\{(M_{t_k^n} - M_{t_{k-1}^n})^4 \mid \mathcal{F}_{t_{k-1}^n}\big\}$$

exists for the topology $\sigma(L^1, L^\infty)$. Moreover,

$$(M)^{(\tilde{4})} \;=\; \sum_{i\geq 1}\mathbf{E}\{X_i^4 \mid \mathcal{F}_{S_i-}\}1_{[S_i,\infty[} \;+\; \sum_{j\geq 1}\tilde{B}^j\,,$$

where the processes $\tilde{B}^j$ are the (continuous) compensators of the $Y_j^4\,1_{[T_j,\infty[}$.

Proof: We take an arbitrary, fixed $Z \in L^\infty$ and denote by $(Y_t)_{t\geq 0}$ a cadlag-modification of $(\mathbf{E}\{Z|\mathcal{F}_t\})_{t\geq 0}$. Then we have

$$\begin{aligned}
&\mathbf{E}\, Z\, (M)_t^{(\tilde{4}),n} \\
&:= \quad \mathbf{E}\, Z \sum_{k=1}^{m(t)} \mathbf{E}\big\{(M_{t_k^n} - M_{t_{k-1}^n})^4 \,|\, \mathcal{F}_{t_{k-1}^n}\big\} \\
&= \quad \mathbf{E} \sum_{k=1}^{m(t)} Y_{t_{k-1}^n} (M_{t_k^n} - M_{t_{k-1}^n})^4 \,.
\end{aligned}$$

As in the proof of theorem 4.1 one can now show that

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{k=1}^{m(t)} Y_{t_{k-1}^n} (M_{t_k^n} - M_{t_{k-1}^n})^4 \\
&\quad = \lim_{n\to\infty} \sum_{k=1}^{m(t)} Y_{t_{k-1}^n} (M^d_{t_k^n} - M^d_{t_{k-1}^n})^4 \\
&\quad = \int_{[0,t]} Y_{s-}\, d(M)_s^{(4)}
\end{aligned}$$

in L^1. The process

$$\widetilde{(M)^{(4)}} = \sum_{i\geq 1} \mathbf{E}\{X_i^4 \,|\, \mathcal{F}_{S_i-}\} 1_{[S_i,\infty[} + \sum_{j\geq 1} \tilde{B}^j$$

is the compensator of $(M)^{(4)}$, and hence

$$\mathbf{E} \int_{[0,t]} Y_{s-}\, d(M)_s^{(4)} = \mathbf{E} \int_{[0,t]} Y_{s-}\, d\widetilde{(M)^{(4)}}_s \,.$$

From

$$\begin{aligned}
&\big|\mathbf{E}\, Z\, (M)_t^{(\tilde{4}),n} - \mathbf{E}\, Z\, \widetilde{(M)^{(4)}}_t \big| \\
&\quad = \big|\mathbf{E}\, Z\, (M)_t^{(\tilde{4}),n} - \mathbf{E} \int_{[0,t]} Y_{s-}\, d\widetilde{(M)^{(4)}}_s \big| \\
&\quad \leq \mathbf{E} \Big| \sum_{k=1}^{m(t)} Y_{t_{k-1}^n} (M_{t_k^n} - M_{t_{k-1}^n})^4 - \int_{[0,t]} Y_{s-}\, d(M)_s^{(4)} \Big|
\end{aligned}$$

we get that

$$\lim_{n\to\infty} (M)_t^{(\tilde{4}),n} = \widetilde{(M)^{(4)}}_t$$

for $\sigma(L^1, L^\infty)$ and the theorem is proved. □

Theorem 4.3 (Existence of $(M)^{(2,\tilde{2})}$) *For every $t \geq 0$,*

$$(M)_t^{(2,\tilde{2})} := \lim_{n\to\infty} \sum_{k=1}^{m(t)} (M_{t_k^n} - M_{t_{k-1}^n})^2 \mathbf{E}\big\{(M_{t_k^n} - M_{t_{k-1}^n})^2 \,|\, \mathcal{F}_{t_{k-1}^n}\big\}$$

exists for the topology $\sigma(L^1, L^\infty)$, *and*

$$(M)^{(2,\tilde{2})} \;=\; \sum_{i\geq 1} X_i^2 \mathbf{E}\{X_i^2 \mid \mathcal{F}_{S_i-}\}\, 1_{[S_i,\infty[}\,.$$

Proof: We set $\bar{M}^p := M - M^p$. Then we have

$$\begin{aligned}
&(M)_t^{(2,\tilde{2}),n}\\
&:= \sum_{k=1}^{m(t)} (M_{t_k^n} - M_{t_{k-1}^n})^2 \mathbf{E}\big\{(M_{t_k^n} - M_{t_{k-1}^n})^2 \mid \mathcal{F}_{t_{k-1}^n}\big\}\\
&= \sum_{k=1}^{m(t)} \big((\bar{M}^p_{t_k^n} - \bar{M}^p_{t_{k-1}^n}) + (M^p_{t_k^n} - M^p_{t_{k-1}^n})\big)^2\\
&\qquad\qquad \cdot \mathbf{E}\big\{\big((\bar{M}^p_{t_k^n} - \bar{M}^p_{t_{k-1}^n}) + (M^p_{t_k^n} - M^p_{t_{k-1}^n})\big)^2 | \mathcal{F}_{t_{k-1}^n}\big\}\\
&= \sum_{k=1}^{m(t)} (M^p_{t_k^n} - M^p_{t_{k-1}^n})^2 \mathbf{E}\big\{ <M^p>_{t_k^n} - <M^p>_{t_{k-1}^n} |\mathcal{F}_{t_{k-1}^n}\big\}\\
&\qquad\qquad + C_1^n + C_2^n + 2C_3^n\,,
\end{aligned}$$

where

$$\begin{aligned}
C_1^n &:= \sum_{k=1}^{m(t)} (\bar{M}^p_{t_k^n} - \bar{M}^p_{t_{k-1}^n})^2 \mathbf{E}\big\{ <M>_{t_k^n} - <M>_{t_{k-1}^n} |\mathcal{F}_{t_{k-1}^n}\big\}\,,\\
C_2^n &:= \sum_{k=1}^{m(t)} (M^p_{t_k^n} - M^p_{t_{k-1}^n})^2 \mathbf{E}\big\{ <\bar{M}^p>_{t_k^n} - <\bar{M}^p>_{t_{k-1}^n} |\mathcal{F}_{t_{k-1}^n}\big\}\,, \text{ and}\\
C_3^n &:= \sum_{k=1}^{m(t)} (\bar{M}^p_{t_k^n} - \bar{M}^p_{t_{k-1}^n})(M^p_{t_k^n} - M^p_{t_{k-1}^n}) \mathbf{E}\big\{ <M>_{t_k^n} - <M>_{t_{k-1}^n} |\mathcal{F}_{t_{k-1}^n}\big\}\,.
\end{aligned}$$

First we prove that $\lim_{n\to\infty} C_i^n = 0$ in L^1 $(i = 1, 2, 3)$. We have

$$\begin{aligned}
&\mathbf{E}\,|C_1^n|\\
&= \mathbf{E}\Big(\sum_{k=1}^{m(t)} \mathbf{E}\big\{ <\bar{M}^p>_{t_k^n} - <\bar{M}^p>_{t_{k-1}^n} |\mathcal{F}_{t_{k-1}^n}\big\}\big(<M>_{t_k^n} - <M>_{t_{k-1}^n} \big)\Big)\\
&\leq \mathbf{E}\Big(\max_k \mathbf{E}\big\{ <\bar{M}^p>_{t_k^n} - <\bar{M}^p>_{t_{k-1}^n} |\mathcal{F}_{t_{k-1}^n}\big\} <M>_t \Big)\\
&\leq \Big[\mathbf{E}\Big(\max_k \big(\mathbf{E}\big\{ <\bar{M}^p>_{t_k^n} - <\bar{M}^p>_{t_{k-1}^n} |\mathcal{F}_{t_{k-1}^n}\big\}\big)^2\Big)\Big]^{\frac{1}{2}} \big[\mathbf{E} <M>_t^2 \big]^{\frac{1}{2}}\,.
\end{aligned}$$

Now

$$\begin{aligned}
&\mathbf{E}\max_k \left(\mathbf{E}\{ <\bar{M}^p>_{t^n_k} - <\bar{M}^p>_{t^n_{k-1}} | \mathcal{F}_{t^n_{k-1}}\}\right)^2 \\
&\leq \mathbf{E}\sum_{k=1}^{m(t)} \left(\mathbf{E}\{ <\bar{M}^p>_{t^n_k} - <\bar{M}^p>_{t^n_{k-1}} | \mathcal{F}_{t^n_{k-1}}\}\right)^2 \\
&\leq \mathbf{E}\sum_{k=1}^{m(t)} \left(<\bar{M}^p>_{t^n_k} - <\bar{M}^p>_{t^n_{k-1}} \right)^2,
\end{aligned}$$

and the last expression tends to zero for $n \to \infty$, since $<\bar{M}^p>$ is continuous. So we have proved that $\lim_{n\to\infty} C^n_1 = 0$ in L^1.

Again using the continuity of $<\bar{M}^p>$, one can prove similarly that also $\lim_{n\to\infty} C^n_2 = 0$ in L^1.

For the proof that $\lim_{n\to\infty} C^n_3 = 0$ we first show that one can reduce the problem to the case where the number of jumps of M on $[0,t]$ is bounded. Then

$$C^n_3 \;=\; D^n_1 \;+\; D^n_2$$

with

$$D^n_1 := \sum_{k=1}^{m(t)} (M^c_{t^n_k} - M^c_{t^n_{k-1}})(M^p_{t^n_k} - M^p_{t^n_{k-1}})\mathbf{E}\{ <M>_{t^n_k} - <M>_{t^n_{k-1}} | \mathcal{F}_{t^n_{k-1}}\}$$

and

$$D^n_2 := \sum_{k=1}^{m(t)} (M^t_{t^n_k} - M^t_{t^n_{k-1}})(M^p_{t^n_k} - M^p_{t^n_{k-1}})\mathbf{E}\{ <M>_{t^n_k} - <M>_{t^n_{k-1}} | \mathcal{F}_{t^n_{k-1}}\}\,.$$

Then $\lim_{n\to\infty} \mathbf{E}D^n_1 = 0$ because of the continuity of M^c, and $\lim_{n\to\infty} \mathbf{E}D^n_2 = 0$, since the finite many jumps of M^t and M^p have pairwise disjoint graphs.

As far we have proved that for the existence of $(M)^{(2,\tilde{2})}$ we only have to prove that

$$(M^p)^{(2,\tilde{2})}_t \;=\; \lim_{n\to\infty} (M^p)^{(2,\tilde{2}),n}_t$$

exists for $\sigma(L^1, L^\infty)$, where

$$(M^p)^{(2,\tilde{2}),n}_t \;:=\; \sum_{k=1}^{m(t)} (M^p_{t^n_k} - M^p_{t^n_{k-1}})^2\, \mathbf{E}\{ <M^p>_{t^n_k} - <M^p>_{t^n_{k-1}} | \mathcal{F}_{t^n_{k-1}}\}\,.$$

Again, one first proves that this problem can be reduced to the case that M^p has only finite many jumps. We omit this proof and assume now that

$$M^p \;=\; \sum_{i=1}^{m} X_i\, 1_{[S_i,\infty[}\,.$$

To prove the assertion for an M^p of the above form, we need the following lemma.

Lemma 4.4 *Let $X = (X_t)_{t\geq 0}$ be an increasing, right continuous, integrable, $\mathbf{F}$-predictable process. For every stopping time T we set*

$$\tilde{X}_T^n \;:=\; \sum_{k\geq 1}\big(\sum_{j=1}^{k}\mathbf{E}\{X_{t_j^n} - Xt_{j-1}^n|\mathcal{F}_{t_{j-1}^n}\}\big)\,1_{]t_{k-1}^n,t_k^n]}(T)\,,\text{ and}$$

$$\bar{X}_T^n \;:=\; \sum_{k\geq 1}\big(\sum_{j=1}^{k-1}\mathbf{E}\{X_{t_j^n} - Xt_{j-1}^n|\mathcal{F}_{t_{j-1}^n}\}\big)\,1_{]t_{k-1}^n,t_k^n]}(T)$$

for every $n\geq 1$. If T is predictable, then

$$\lim_{n\to\infty}(\tilde{X}_T^n \,-\, \bar{X}_T^n) \;=\; \triangle X_T$$

for the topoloy $\sigma(L^1, L^\infty)$.

Proof of the lemma: It follows from theorem 3.2 that $\lim_{n\to\infty}\tilde{X}_T^n = X_T$ for $\sigma(L^1, L^\infty)$. Hence it remains to prove that $\lim_{n\to\infty}\bar{X}_T^n = X_{T-}$ for $\sigma(L^1, L^\infty)$.

Let $(T_i)_{i\geq 1}$ be an announcing sequence for T. From theorem 3.2 we have $\lim_{n\to\infty}\tilde{X}_{T_i}^n = X_{T_i}$ for $\sigma(L^1, L^\infty)$ uniformly in $i\in\mathbf{N}$. Moreover, the $\sigma(L^1, L^\infty)$-convergence of $(\tilde{X}_{T_i}^n)_{n\geq 1}$ implies (cf. [4],p.20) that for every fixed non-negative $Z\in L^\infty$ the sequence $(Z\,\tilde{X}_{T_i}^n)_{n\geq 1}$ is uniformly integrable for every $i\geq 1$. Hence for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, i)$ such that

$$\mathbf{E}\,1_{[T-T_i\leq\delta]}\,Z\,\tilde{X}_{T_i}^n \;<\; \varepsilon\,.$$

Now, for every $\omega\in[T-T_i>\delta]$ there exists an $n(\delta)$ such that

$$\tilde{X}_{T_i(\omega)}^n(\omega) \;\leq\; \bar{X}_{T(\omega)}^n(\omega)$$

for $n\geq n(\delta)$. Therefore,

$$\mathbf{E}\,Z\,\tilde{X}_{T_i}^n \;\leq\; \mathbf{E}\,1_{[T-T_i>\,\delta]}\,Z\,\tilde{X}_{T_i}^n \,+\, \varepsilon \;\leq\; \mathbf{E}\,Z\,\bar{X}_T^n \,+\, \varepsilon$$

for every $n\geq n(\delta)$. It follows that

$$\mathbf{E}\,Z\,X_{T_i} \;=\; \lim_{n\to\infty}\mathbf{E}\,Z\,\tilde{X}_{T_i}^n \;\leq \limsup_{n\to\infty}\mathbf{E}\,Z\,\bar{X}_T^n \,+\, \varepsilon\,.$$

On the other side, we have for every fixed $n\geq 1$

$$\mathbf{E}\,Z\,\tilde{X}_T^n \;=\; \lim_{i\to\infty}\mathbf{E}\,Z\,\tilde{X}_{T_i}^n\,,$$

and hence

$$\mathbf{E}\,Z\,\bar{X}_T^n \;\leq\; \mathbf{E}\,Z\,\tilde{X}_T^n \;\leq\; \mathbf{E}\,Z\,\tilde{X}_{T_i}^n \,+\, \varepsilon$$

for all $i \geq i(n,\varepsilon)$. Since

$$\lim_{n\to\infty} \mathbf{E}\, Z\, \tilde{X}^n_{T_i} \;=\; \mathbf{E}\, Z\, X_{T_i}$$

uniformly in $i \geq 1$, there exists an $n(\varepsilon)$ such that for $n \geq n(\varepsilon)$ and for an $i(n)$,

$$\mathbf{E}\, Z\, \tilde{X}^n_{T_{i(n)}} \;\leq\; \mathbf{E}\, Z\, X_{T_{i(n)}} \;+\; \varepsilon\,.$$

Hence we have

$$\limsup_{n\to\infty} \mathbf{E}\, Z\, \bar{X}^n_T \;\leq\; \limsup_{n\to\infty} \mathbf{E}\, Z\, X_{T_{i(n)}} \;+\; 2\,\varepsilon\,.$$

So far we have proved

$$\begin{aligned}
\mathbf{E}\, Z\, X_{T-} \;&=\; \lim_{i\to\infty} \mathbf{E}\, Z\, X^n_{T_i} \\
&\leq\; \liminf_{n\to\infty} \mathbf{E}\, Z\, \bar{X}^n_T \;+\; \varepsilon \\
&\leq\; \limsup_{n\to\infty} \mathbf{E}\, Z\, \bar{X}^n_T \;+\; \varepsilon \\
&\leq\; \limsup_{n\to\infty} \mathbf{E}\, Z\, X_{T_{i(n)}} \;+\; 3\,\varepsilon\,,
\end{aligned}$$

or

$$\begin{aligned}
\mathbf{E}\, Z\, X_{T-} \;-\; \varepsilon \;&\leq\; \liminf_{n\to\infty} \mathbf{E}\, Z\, \bar{X}^n_T \\
&\leq\; \limsup_{n\to\infty} \mathbf{E}\, Z\, \bar{X}^n_T \\
&\leq\; \mathbf{E}\, Z\, X_{T-} \;+\; 2\,\varepsilon\,.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrarily chosen,

$$\lim_{n\to\infty} \mathbf{E}\, Z\, \bar{X}^n_T \;=\; \mathbf{E}\, Z\, X_{T-}$$

and the lemma is proved. □

Now we complete the proof of theorem 4.3 using the above lemma. We set

$$M^j \;=\; X_j\, 1_{[S_j,\infty[}\,.$$

Then $M^p = \sum_{j=1}^{m(t)} M^j$. For a fixed $j = 1, \cdots, m$ we set $T := S_j$ and $X := <M^p>$. Then

$$\begin{aligned}
&\sum_{k=1}^{m(t)} (M^j_{t^n_k} - M^j_{t^n_{k-1}})^2\, \mathbf{E}\big\{ <M^p>_{t^n_k} - <M^p>_{t^n_{k-1}} |\mathcal{F}_{t^n_{k-1}}\big\} \\
&\quad=\; \sum_{k=1}^{m(t)} X_j^2\, 1_{]t^n_{k-1},t^n_k]}(T)\mathbf{E}\{X_{t^n_k} - X t^n_{k-1}|\mathcal{F}_{t^n_{k-1}}\} \\
&\quad=\; X_j^2\, (\tilde{X}^n_T \;-\; \bar{X}^n_T)\, 1_{[T\leq t]}\,.
\end{aligned}$$

Suppose first that the jump height X_j is bounded. Then we get from the lemma that

$$\begin{aligned}\lim_{n\to\infty} X_j^2\,(\tilde{X}_T^n \;-\; \bar{X}_T^n)\,1_{[T\le t]} \;&=\; X_j^2 \triangle X_T\,1_{[T\le t]} \\ =\;\; X_j^2 \triangle <M^p>_{S_j}\;1_{[S_j\le t]} \;&=\; X_j^2\,\mathbf{E}\{X_j^2|\mathcal{F}_{S_j-}\}\,1_{[S_j\le t]}\end{aligned}$$

for $\sigma(L^1,L^\infty)$. By a uniform integrability argument, this limit relation also holds, if X_j is not necessarily bounded, and we have thus proved that for $\sigma(L^1,L^\infty)$

$$\lim_{n\to\infty}(M^p)_t^{(2,\tilde{2}),n} \;=\; \sum_{i=1}^m X_i^2\,\mathbf{E}\{X_i^2|\mathcal{F}_{S_i-}\}\,1_{[S_i\le t]}\,.$$

This finishes the proof of the theorem. □

Theorem 4.5 (Existence of $(M)^{(\tilde{2},\tilde{2})}$) *For every $t\ge 0$,*

$$(M)_t^{(\tilde{2},\tilde{2})} \;:=\; \lim_{n\to\infty}\sum_{k=1}^{m(t)}\Big(E\big\{(M_{t_k^n}-M_{t_{k-1}^n})^2\mid\mathcal{F}_{t_{k-1}^n}\big\}\Big)^2$$

exists for the topology $\sigma(L^1,L^\infty)$ and

$$(M)^{(\tilde{2},\tilde{2})} \;=\; \sum_{i\ge 1}\big(\mathbf{E}\{X_i^2|\mathcal{F}_{S_i-}\}\big)^2\,1_{[S_i,\infty[}\,.$$

Proof: With similar arguments as in the proof of theorem 4.3 one first shows that the $\sigma(L^1,L^\infty)$-convergence of

$$(M)_t^{(\tilde{2},\tilde{2}),n} \;:=\; \sum_{k=1}^{m(t)}\Big(E\big\{(M_{t_k^n}-M_{t_{k-1}^n})^2\mid\mathcal{F}_{t_{k-1}^n}\big\}\Big)^2$$

can be reduced to the problem of the $\sigma(L^1,L^\infty)$-convergence of

$$(M^p)_t^{(\tilde{2},\tilde{2}),n} \;:=\; \sum_{k=1}^{m(t)}\Big(E\big\{(M^p_{t_k^n}-M^p_{t_{k-1}^n})^2\mid\mathcal{F}_{t_{k-1}^n}\big\}\Big)^2,$$

and that one can even assume that M^p has only a finite number of jumps, i.e. we assume that

$$M^p \;=\; \sum_{i=1}^m X_i\,1_{[S_i,\infty[}\,.$$

Now let Z be a fixed non-negative element of $L^\infty(\Omega)$ and denote by $(Y_t)_{t\ge 0}$ a cadlag-modification of the martingale $(\mathbf{E}\{Z|\mathcal{F}_t\})_{t\ge 0}$. Then we have

$$\begin{aligned}&\mathbf{E}\,Z\,(M^p)_t^{(\tilde{2},\tilde{2}),n}\\ &=\;\mathbf{E}\,Z\sum_{k=1}^{m(t)}\Big(E\big\{<M^p>_{t_k^n}-<M^p>_{t_{k-1}^n}\mid\mathcal{F}_{t_{k-1}^n}\big\}\Big)^2\\ &=\;\mathbf{E}\sum_{k=1}^{m(t)}Y_{t_{k-1}^n}\big(<M^p>_{t_k^n}-<M^p>_{t_{k-1}^n}\big)\mathbf{E}\big\{<M^p>_{t_k^n}-<M^p>_{t_{k-1}^n}\mid\mathcal{F}_{t_{k-1}^n}\big\}\,.\end{aligned}$$

Since

$$<M^p> = \sum_{i=1}^{m} \mathbf{E}\{X_i^2|\mathcal{F}_{S_i-}\}\, 1_{[S_i,\infty[}\,,$$

we have to prove for the convergence of the sequence $(\mathbf{E}\, Z\,(M^p)_t^{(\tilde{2},\tilde{2}),n})_{n\geq 1}$ that for every fixed $i = 1, \cdots, m$ the sequence

$$\left(\mathbf{E}\sum_{k=1}^{m(t)} Y_{t_{k-1}^n}\mathbf{E}\{X_i^2|\mathcal{F}_{S_i-}\}\, 1_{]t_{k-1}^n,t_k^n]}(S_i)\mathbf{E}\big\{ <M^p>_{t_k^n} - <M^p>_{t_{k-1}^n} \mid \mathcal{F}_{t_{k-1}^n}\big\}\right)_{n\geq 1}$$

converges. As in the proof of the last theorem we show the convergence of that sequence by proving that the sequence

$$\left(\sum_{k=1}^{m(t)} Y_{t_{k-1}^n}\, 1_{]t_{k-1}^n,t_k^n]}(S_i)\mathbf{E}\big\{ <M^p>_{t_k^n} - <M^p>_{t_{k-1}^n} \mid \mathcal{F}_{t_{k-1}^n}\big\}\right)_{n\geq 1}$$

converges for $\sigma(L^1, L^\infty)$.

We set $T := S_i$ and $X :=<M^p>$ and define for every $t \geq 0$,

$$\tilde{Y}_t^n := \sum_{k\geq 1}\Big(\sum_{j=1}^{k} Y_{t_{j-1}^n}\mathbf{E}\{X_{t_j^n} - X_{t_{j-1}^n}|\,\mathcal{F}_{t_{j-1}^n}\}\Big)\, 1_{]t_{k-1}^n,t_k^n]}(t)\,.$$

Then we assert that the sequence $(\tilde{Y}_t^n)_{n\geq 1}$ converges for $\sigma(L^1, L^\infty)$. We take an arbitrary $V \in L^\infty$ and a cadlag-modification $(U_t)_{t\geq 0}$ of $(\mathbf{E}\{V|\mathcal{F}_t\})_{t\geq 0}$. Then

$$\begin{aligned}
&\lim_{n\to\infty}\mathbf{E}\, V\,\tilde{Y}_t^n \\
&\quad= \lim_{n\to\infty}\ \mathbf{E}\sum_{k=1}^{m(t)} U_{t_{k-1}^n} Y_{t_{k-1}^n}\mathbf{E}\{X_{t_k^n} - X_{t_{k-1}^n}|\,\mathcal{F}_{t_{k-1}^n}\} \\
&\quad= \lim_{n\to\infty}\ \mathbf{E}\sum_{k=1}^{m(t)} U_{t_{k-1}^n} Y_{t_{k-1}^n}(X_{t_k^n} - X_{t_{k-1}^n}) \\
&\quad= \mathbf{E}\int_0^t U_{s-}\, Y_{s-}\, dX_s \\
&\quad= \mathbf{E}\int_0^t U_{s-}\, dW_s \\
&\qquad\text{(where } W \text{ is the predictable process given by } W_t = \int_0^t Y_{s-} dX_s) \\
&\quad= \mathbf{E}\, U_t \int_0^t Y_{s-}\, dX_s \\
&\quad= \mathbf{E}\, V \int_0^t Y_{s-}\, dX_s\,,
\end{aligned}$$

and we have proved that

$$\lim_{n\to\infty} \tilde{Y}^n_t \;=\; \int_0^t Y_{s-}\, dX_s$$

for the $\sigma(L^1, L^\infty)$-topology. more general, if T is a stopping time such that $X_T \in L^1$, then

$$\lim_{n\to\infty} \tilde{Y}^n_S \;=\; \int_0^S Y_{s-}\, dX_s$$

for $\sigma(L^1, L^\infty)$ uniformly on the set of all stopping times S with $S \le T$.

Now we define for every $n \ge 1$ and $t \ge 0$,

$$\bar{Y}^n_t \;:=\; \sum_{k\ge 1}\Big(\sum_{j=1}^{k-1} Y_{t^n_{j-1}}\mathbf{E}\{X_{t^n_j} - X_{t^n_{j-1}} \,|\, \mathcal{F}_{t^n_{j-1}}\}\Big)\, 1_{]t^n_{k-1},t^n_k]}(t)\,.$$

Then one can prove exactly as in the proof of lemma 4.4 that

$$\lim_{n\to\infty} (\tilde{Y}^n_T \;-\; \bar{Y}^n_T) \;=\; \int_0^T Y_{s-}\, dX_s \;-\; \int_0^{T-} Y_{s-}\, dX_s \;=\; Y_{T-}\triangle X_T$$

for $\sigma(L^1, L^\infty)$, using the predictability of T. So we have proved that

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{k=1}^{m(t)} Y_{t^n_{k-1}}\, 1_{]t^n_{k-1},t^n_k]}(S_i)\mathbf{E}\big\{ <M^p>_{t^n_k} - <M^p>_{t^n_{k-1}} \;|\, \mathcal{F}_{t^n_{k-1}}\big\} \\
&\quad = Y_{S_i-}\triangle <M^p>_{S_i}\; 1_{[S_i\le t]} \\
&\quad = Y_{S_i-}\,\mathbf{E}\{X_i^2|\mathcal{F}_{S_i-}\}\, 1_{[S_i\le t]}
\end{aligned}$$

for $\sigma(L^1, L^\infty)$. Assuming first that the jump X_i is bounded, the just proved relation implies

$$\begin{aligned}
&\lim_{n\to\infty}\mathbf{E}\bigg(\sum_{k=1}^{m(t)} Y_{t^n_{k-1}}\mathbf{E}\{X_i^2|\mathcal{F}_{S_i-}\}\, 1_{]t^n_{k-1},t^n_k]}(S_i)\cdot\mathbf{E}\big\{ <M^p>_{t^n_k} - <M^p>_{t^n_{k-1}} \;|\, \mathcal{F}_{t^n_{k-1}}\big\}\bigg) \\
&\quad = \mathbf{E}\Big(Y_{S_i-}\,\big(\mathbf{E}\{X_i^2|\mathcal{F}_{S_i-}\}\big)^2\, 1_{[S_i\le t]}\Big) \\
&\quad = \mathbf{E}\Big(Z\,\big(\mathbf{E}\{X_i^2|\mathcal{F}_{S_i-}\}\big)^2\, 1_{[S_i\le t]}\Big)\,.
\end{aligned}$$

Again, one can prove by a uniform integrability argument that the last limit relation also holds for general X_i, and the theorem is proved. □

Theorem 4.6 (Existence of $(M)^{(1,\tilde{3})}$) *For every $t \ge 0$,*

$$(M)^{(1,\tilde{3})}_t \;:=\; \lim_{n\to\infty}\sum_{k=1}^{m(t)} (M_{t^n_k} - M_{t^n_{k-1}})\,\mathbf{E}\big\{(M_{t^n_k} - M_{t^n_{k-1}})^3 \,|\, \mathcal{F}_{t^n_{k-1}}\big\}$$

exists for the $\sigma(L^1, L^\infty)$-topology, and

$$(M)^{(1,\tilde{3})} \;=\; \sum_{i\ge 1} X_i\mathbf{E}\{X_i^3 \,|\, \mathcal{F}_{S_i-}\}\, 1_{[S_i,\infty[}\,.$$

Proof: Partially,we just indicate the ideas of the proof, since the arguments are often similar to those in the proofs of the foregoing results.
(1) First one shows that

$$(M)_t^{(3)} \;:=\; \lim_{n\to\infty} \sum_{k=1}^{m(t)} (M_{t_k^n} - M_{t_{k-1}^n})^3$$

exists in L^1 and that

$$(M)^{(3)} \;=\; \sum_{i\geq 1} X_i^3 \, 1_{[S_i,\infty[} \;+\; \sum_{j\geq 1} Y_j^3 \, 1_{[T_j,\infty[} \,.$$

The proof is similar to the proof of theorem 4.1. Now let $\tilde{C}^j$ denote the continuous compensator of $Y_j^3 \, 1_{[T_j,\infty[}$. Then the compensator of $(M)^{(3)}$ is given by

$$(M)^{(\tilde{3})} := \sum_{i\geq 1} \mathbf{E}\{X_i^3 | \mathcal{F}_{S_i -}\} \, 1_{[S_i,\infty[} \;+\; \sum_{j\geq 1} \tilde{C}^j \,.$$

(2) For every $n \geq 1$ and $t \geq 0$ we set

$$(M)_t^{(\tilde{3}),n} \;:= \sum_{k=1}^{m(t)} \mathbf{E}\big\{(M_{t_k^n} - M_{t_{k-1}^n})^3 \mid \mathcal{F}_{t_{k-1}^n}\big\} \,.$$

Let $Z \in L^\infty$ be given and denote again by $(Y_t)_{t\geq 0}$ a cadlag-modification of $(\mathbf{E}\{Z|\mathcal{F}_t\})_{t\geq 0}$. Then

$$\mathbf{E}\, Z \, (M)_t^{(\tilde{3}),n} \;=\; \mathbf{E} \sum_{k=1}^{m(t)} Y_{t_{k-1}^n} (M_{t_k^n} - M_{t_{k-1}^n})^3 \,.$$

As in the proof of theorem 4.2 it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \mathbf{E}\, Z \, (M)_t^{(\tilde{3}),n} \\
&\quad= \lim_{n\to\infty} \mathbf{E}\, Z \, (M^d)_t^{(\tilde{3}),n} \\
&\quad= \mathbf{E} \int_0^t Y_{s-} \, d(M^d)_s^{(3)} \\
&\quad= \mathbf{E} \int_0^t Y_{s-} \, d(M^d)_s^{(\tilde{3})} \\
&\quad= \mathbf{E}\, Z \, (M^d)_t^{(\tilde{3})} \,,
\end{aligned}$$

and hence we have proved that

$$\lim_{n\to\infty} (M)_t^{(\tilde{3}),n} \;=\; (M)_t^{(\tilde{3})} \;=\; \lim_{n\to\infty} (M^d)_t^{(\tilde{3}),n} \;=\; (M^d)_t^{(\tilde{3})}$$

for $\sigma(L^1, L^\infty)$.

Consider now the sequence $((M)_t^{(1,\tilde{3}),n})_{n\geq 1}$ defined by

$$(M)_t^{(1,\tilde{3}),n} := \sum_{k=1}^{m(t)} (M_{t_k^n} - M_{t_{k-1}^n}) \mathbf{E}\big\{(M_{t_k^n} - M_{t_{k-1}^n})^3 \mid \mathcal{F}_{t_{k-1}^n}\big\}\,.$$

It is not difficult to see that the $\sigma(L^1, L^\infty)$-convergence of that sequence is equivalent to the $\sigma(L^1, L^\infty)$-convergence of the corresponding sequence $((M^d)_t^{(1,\tilde{3}),n})_{n\geq 1}$, where M is replaced by M^d. For analyzing the limit behaviour of the last sequence, let us first assume that M^d is of the simple form

$$M^d = X\, 1_{[S,\infty[} + Y\, 1_{[T,\infty[}$$

with $X, Y \in L^4$ and S, T stopping times. We get

$$\begin{aligned}
&(M^d)_t^{(1,\tilde{3}),n} \\
&= \sum_{k=1}^{m(t)} \Big[\Big(X\, 1_{]t_{k-1}^n, t_k^n]}(S) + Y\, 1_{]t_{k-1}^n, t_k^n]}(T) \Big) \\
&\qquad\qquad \cdot \mathbf{E}\big\{ X^3\, 1_{]t_{k-1}^n, t_k^n]}(S) + Y^3\, 1_{]t_{k-1}^n, t_k^n]}(T) \mid \mathcal{F}_{t_{k-1}^n} \big\} \Big] \\
&\quad + A^n\,,
\end{aligned}$$

where

$$\begin{aligned}
A^n := \sum_{k=1}^{m(t)} \Big[&\Big(X\, 1_{]t_{k-1}^n, t_k^n]}(S) + Y\, 1_{]t_{k-1}^n, t_k^n]}(T) \Big) \\
&\cdot \mathbf{E}\big\{ 3(XY^2 + X^2Y) 1_{]t_{k-1}^n, t_k^n]}(S)\, 1_{]t_{k-1}^n, t_k^n]}(T) \mid \mathcal{F}_{t_{k-1}^n} \big\} \Big]\,.
\end{aligned}$$

For A^n we get (with $Z := 3(XY^2 + X^2Y)$)

$$\begin{aligned}
&\mathbf{E}\,|A^n| \\
&\leq \mathbf{E} \sum_{k=1}^{m(t)} \Big[\Big(|X|\, 1_{]t_{k-1}^n, t_k^n]}(S) + |Y|\, 1_{]t_{k-1}^n, t_k^n]}(T) \Big) \\
&\qquad\qquad \cdot \mathbf{E}\big\{ |Z|\, 1_{]t_{k-1}^n, t_k^n]}(S)\, 1_{]t_{k-1}^n, t_k^n]}(T) \mid \mathcal{F}_{t_{k-1}^n} \big\} \Big] \\
&= \mathbf{E} \sum_{k=1}^{m(t)} \Big[\mathbf{E}\Big\{ |X|\, 1_{]t_{k-1}^n, t_k^n]}(S) + |Y|\, 1_{]t_{k-1}^n, t_k^n]}(T) \mid \mathcal{F}_{t_{k-1}^n} \Big\} \\
&\qquad\qquad \cdot \Big(|Z|\, 1_{]t_{k-1}^n, t_k^n]}(S)\, 1_{]t_{k-1}^n, t_k^n]}(T) \Big) \Big]\,.
\end{aligned}$$

If the stopping times S and T have disjoint graphs, it is not difficult to conclude from the above inequality that

$$\lim_{n\to\infty} \mathbf{E}\,|A^n| = 0\,.$$

Suppose now that S is predictable and T is totally inaccessible and denote by $\tilde{D}^i$ $(i = 1,2)$ the continuous compensators of $|Y|\,1_{T,\infty[}$ and $|Y^3|\,1_{T,\infty[}$ resp.. Then we have

$$\begin{aligned}(M^d)_t^{(1,\tilde{3}),n} \;-\; A^n \\ &= \sum_{k=1}^{m(t)} X\,1_{]t^n_{k-1},t^n_k]}(S)\mathbf{E}\big\{X^3\,1_{]t^n_{k-1},t^n_k]}(S)\mid \mathcal{F}_{t^n_{k-1}}\big\} \\ &\quad + B^n \;+\; C^n\end{aligned}$$

with

$$B^n \;:=\; \sum_{k=1}^{m(t)} \Big(Y\,1_{]t^n_{k-1},t^n_k]}(T)\mathbf{E}\big\{X^3\,1_{]t^n_{k-1},t^n_k]}(S) \;+\; Y^3\,1_{]t^n_{k-1},t^n_k]}(T)\mid \mathcal{F}_{t^n_{k-1}}\big\}\Big)$$

and

$$C^n \;:=\; \sum_{k=1}^{m(t)} X\,1_{]t^n_{k-1},t^n_k]}(S)\mathbf{E}\big\{Y^3\,1_{]t^n_{k-1},t^n_k]}(S)\mid \mathcal{F}_{t^n_{k-1}}\big\}\,.$$

We get

$$\begin{aligned}&\mathbf{E}\,|B^n| \\ &\leq\; \mathbf{E}\sum_{k=1}^{m(t)}\Big(|Y|\,1_{]t^n_{k-1},t^n_k]}(T) \\ &\qquad\qquad \cdot\, \mathbf{E}\big\{|X^3|\,1_{]t^n_{k-1},t^n_k]}(S) \;+\; |Y^3|\,1_{]t^n_{k-1},t^n_k]}(T)\mid \mathcal{F}_{t^n_{k-1}}\big\}\Big) \\ &=\; \mathbf{E}\sum_{k=1}^{m(t)}\Big(|Y|\,1_{]t^n_{k-1},t^n_k]}(T) \\ &\qquad\qquad \cdot\, \mathbf{E}\Big\{\mathbf{E}\big\{|X^3|\big|\,\mathcal{F}_{S-}\big\}\,1_{]t^n_{k-1},t^n_k]}(S) \;+\; (\tilde{D}^2_{t^n_k} \;-\; \tilde{D}^2_{t^n_{k-1}})\,\Big|\,\mathcal{F}_{t^n_{k-1}}\Big\}\Big) \\ &=\; \mathbf{E}\sum_{k=1}^{m(t)}\Big(\mathbf{E}\big\{|Y|\,1_{]t^n_{k-1},t^n_k]}(T)\,\big|\,\mathcal{F}_{t^n_{k-1}}\big\} \\ &\qquad\qquad \cdot\, \Big[\mathbf{E}\big\{|X^3|\big|\,\mathcal{F}_{S-}\big\}\,1_{]t^n_{k-1},t^n_k]}(S) \;+\; (\tilde{D}^2_{t^n_k} \;-\; \tilde{D}^2_{t^n_{k-1}})\Big]\Big) \\ &=\; \mathbf{E}\sum_{k=1}^{m(t)}\Big(\mathbf{E}\big\{(\tilde{D}^1_{t^n_k} \;-\; \tilde{D}^1_{t^n_{k-1}})\big|\,\mathcal{F}_{t^n_{k-1}}\big\} \\ &\qquad\qquad \cdot\, \Big[\mathbf{E}\big\{|X^3|\big|\,\mathcal{F}_{S-}\big\}\,1_{]t^n_{k-1},t^n_k]}(S) \;+\; (\tilde{D}^2_{t^n_k} \;-\; \tilde{D}^2_{t^n_{k-1}})\Big]\Big)\,.\end{aligned}$$

Using the continuity of $\tilde{D}^1$, it is now easy to show that

$$\lim_{n\to\infty}\,\mathbf{E}\,|B^n| \;=\; 0\,.$$

Similarly, one can show by using the continuity of $\tilde{D}^2$ that

$$\lim_{n\to\infty}\,\mathbf{E}\,|C^n| \;=\; 0\,.$$

We assumed that M^d had just two jumps mainly to avoid overburdening of notation. If M^d has more than two jumps, similar arguments as above give that the convergence of the sequence $((M^d)_t^{(1,\tilde{3}),n})_{n\geq 1}$ is equivalent to the convergence of the sequence

$$\Big(\sum_{k=1}^{m(t)} \big(\sum_{i\geq 1} X_i \, 1_{]t_{k-1}^n, t_k^n]}(S_i)\big)\mathbf{E}\big\{\sum_{i\geq 1} X_i^3 \, 1_{]t_{k-1}^n, t_k^n]}(S_i) \mid \mathcal{F}_{t_{k-1}^n}\big\}\Big)_{n\geq 1} .$$

Now we can proceed as in the proof of theorem 4.3 (cf. especially lemma 4.4) to obtain that the last sequence is convergent for $\sigma(L^1, L^\infty)$ and has the limit

$$\sum_{i\geq 1} X_i \, \mathbf{E}\{X_i^3 | \mathcal{F}_{S_i -}\} \, 1_{[S_i \leq t]} .$$

This finishes the proof of the theorem. □

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