Rostock. Math. Kolloq. **56**, 39–48 (2002)

Subject Classification (AMS)
35J10,35J60,35J30

Nasreddine Megrez

A Nonlinear Elliptic Eigenvalue Problem in Unbounded Domain

KEY WORDS. Nonlinear eigenvalue problem, Schrödinger equation, P-Laplacian, Unbounded domain.

1 Introduction and main result

We find a localization of λ such that the problem

$$(P_\lambda): \quad \begin{cases} -\mathbf{\Delta}_p u + V(x)|u|^{p-2}u = \lambda f(x,u) \\ u_{|\partial\Omega} = 0 \\ \lim\limits_{|x|\longrightarrow\infty} u(x) = 0 \end{cases}$$

has a solution, where $\Omega \subset \mathbb{R}^N$ is an unbounded domain, $N > p \geq 2$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and $V \in L^p_{loc}(\Omega)$ is a continuous potential on Ω satisfying

$$\liminf_{|x|\longrightarrow\infty} V(x) \geq \min_{x\in\Omega} V(x) > 0.$$

Several authors for example P. Felmer, M. Del Pino [1] and P. H. Rabinowitz [4] have studied the problem (P_λ) when $p = 2$ and $\lambda = 1$ under the following assumption:

$$H): \qquad \exists\theta > 2 \;/\; 0 \leq \theta F(s) < f(s)s, \text{ where } F(s) = \int_0^s f(t)dt.$$

They proved that the Palais-Smale sequence given by the mountain-pass lemma for the functional $J(u) = \frac{1}{2}\|u\|_H^2 - \int_\Omega F(u)dx$, has a convergent subsequence, where the limit is a weak solution of the problem.

In [5], I. Schindler has studied the problem (P_λ) when $p = 2$ and $\lim\limits_{|x|\to\infty} V(x) = +\infty$, without assumption $H)$. He gave an interval I such that (P_λ) has a solution for almost every λ satisfying $\frac{1}{\lambda} \in I$.

In this paper, we will follow the ideas in [5], and prove two existence theorems under weaker assumptions on the potential V. The first result uses conditions on V similar to those in [1], the second uses conditions similar to those in [3].

We denote by $\mathcal{B}$ the Banach space defined by the closure of $C_0^\infty(\Omega)$ under the norm

$$\|u\|_{\mathcal{B}} := \left(\int_\Omega |\nabla u(x)|^p + V(x)|u(x)|^p dx\right)^{\frac{1}{p}}.$$

We use the following notations:

- $F(x,s) := \int_0^s f(x,t)dt.$
- $S_t := \{u \in \mathcal{B} : \ \|u\|_{\mathcal{B}}^p = t\}$.
- $g(u) := \int_\Omega F(x,u)dx$, and $\gamma(t) := \sup_{u \in S_t} g(u)$.
- $I := \left(p \inf_{t \neq s} \frac{\gamma(t)-\gamma(s)}{t-s}, p \sup_{t \neq s} \frac{\gamma(t)-\gamma(s)}{t-s}\right)$.

We will use the following assumptions:

H1) There exist a bounded subset K of Ω, and a real α such that

$$\forall x \in \Omega \setminus K, \ \forall s \in \mathbb{R} : \ f(x,s)s \leq \alpha V(x)|s|^p.$$

H2) $\limsup_{s \longrightarrow 0} \frac{f(x,s)}{|s|^{p-1}} = 0.$

H3) $\lim_{|s| \longrightarrow \infty} \frac{f(x,s)}{|s|^{p^*-1}} = 0$ uniformly in x.

H4) V belongs to the reverse Hölder class A_p.

We recall that a nonnegative locally L^q-integrable function V on $\mathbb{R}^N$ is said to belong to the reverse Hölder class A_q $(1 < q < \infty)$ if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|}\int_B |V(x)|^q dx\right)^{\frac{1}{q}} \leq C\left(\frac{1}{|B|}\int_B |V(x)|dx\right) \tag{1}$$

holds for every ball B in $\mathbb{R}^N$; and we say that $V \in A_\infty$ if for any α, $0 < \alpha < 1$, there exists a β, $0 < \beta < 1$ such that for all balls $B \subset \mathbb{R}^N$ and all subsets $E \subset B$,

$$|E| \geq \alpha|B| \Longrightarrow \int_E V(x)dx \geq \beta \int_B V(x)dx. \tag{2}$$

Note that if $V \in A_q$ then $V \in A_\infty$; and if $V \in A_\infty$, then there exists a q, $1 < q < \infty$ such that $V \in A_q$.

H5) $\lim_{|x|\to\infty} m(x,V) = +\infty$, where $m(x,V)$ is an auxiliary function defined by:

$$\frac{1}{m(x,V)} := \sup\left\{ r > 0,\ \frac{1}{r^{N-p}} \int_{B(x,r)} V(y)dy \leq 1 \right\}.$$

$B(x,r)$ is the ball of $\mathbb{R}^N$ of center x and radius r.

The function $m(x,V)$ was introduced by Z. Shen in [8] for $p = 2$, to study the Neumann problem for the operator $-\boldsymbol{\Delta} + V(x)$ on the domain above a Lipschitzian graph. Recently, K. Kurata [3] has used this function to prove the existence of least energy solution for a Schrödinger equation with magnetic potential.

Note that $0 < m(x,V) < +\infty\ \forall x \in \mathbb{R}^N$, and if $r = \dfrac{1}{m(x,V)}$, then $\dfrac{1}{r^{N-p}} \displaystyle\int_{B(x,r)} V(y)dy = 1$.

The problem (P_λ) is equivalent to finding the values of $\rho = \dfrac{1}{\lambda}$ for which there exists u such that

$$g'(u) = \rho \|u\|_B^{p-2} u\,. \tag{3}$$

Let $G_\rho(u) := \dfrac{\rho}{p}\|u\|_{\mathcal{B}}^p - g(u)$.

Our main results are:

Theorem 1.1 *If* **H1** *and* **H3** *hold, then for almost every* λ *satisfying* $\dfrac{1}{\lambda} \in I \cap (\alpha, +\infty)$ *the problem* (P_λ) *has a solution in* $\mathcal{B}\setminus\{0\}$.

If each critical sequence of G_ρ *is bounded, then* (P_λ) *has a solution for every* λ *satisfying* $\dfrac{1}{\lambda} \in I \cap (\alpha, +\infty)$.

Theorem 1.2 *If* **H2**–**H5** *hold, then for almost every* λ *satisfying* $\frac{1}{\lambda} \in I$ *the problem* (P_λ) *has a solution in* $\mathcal{B}\setminus\{0\}$.

If each critical sequence of G_ρ *is bounded, then* (P_λ) *has a solution for every* λ *satisfying* $\frac{1}{\lambda} \in I$.

2 Mountain pass and impasse lemma

We use a modification of the usual Palais-Smale condition:

Definition 2.1 *We say that* G_ρ *satisfies the property (P) if any sequence* $(u_n) \subset \mathcal{B}$ *satisfying*

$$\|u_n\|_{\mathcal{B}}^p \xrightarrow[n\to\infty]{} t \neq 0 \tag{4}$$

$$G'_\rho(u_n) \xrightarrow[n\to\infty]{} 0 \tag{5}$$

has a converging subsequence.

Remark 2.1 If a sequence $(u_n)_n$ satisfying (5) and

$$G_\rho(u_n) \xrightarrow[n\to\infty]{} b \neq 0 \tag{6}$$

is bounded, then modulo a subsequence we have (4).

We recall mountain pass results due to K. Tintarev [11].

Definition 2.2 *We say that G_ρ has the mountain-pass geometry if:*

- $G_\rho(0) = 0$.
- $\exists t_0 > 0,\ \delta > 0$ *such that for* $\|u\|_{\mathcal{B}}^p = t_0,\ G_\rho(u) > \delta$.
- $\exists e \in \mathcal{B} \setminus \{0\},\ \|e\|_{\mathcal{B}}^p > t_0$ *such that* $G_\rho(e) \leq 0$.

We have the following mountain impasse lemma:

Lemma 2.1 *Assume that G_{ρ_0} has the mountain-pass geometry and satisfies the property (P). Then either there exists $u_0 \in \mathcal{B} \setminus \{0\}$ such that $G'_\rho(u_0) = 0$, or there exists a sequence $(u_n, h_n) \in \mathcal{B} \setminus \{0\} \times \mathbb{R}$ such that:*

$$0 < h_n \longrightarrow 0$$

$$\|u_n\|_{\mathcal{B}}^p = t_n \nearrow \infty$$

$$G_{\rho_0}(u_n) = c(t_n) \searrow c$$

$$G'_{\rho_0+h_n}(u_n) = 0.$$

Using results from [11] we have the following lemma [1]:

Lemma 2.2 *Let $\rho_0 \in \mathbb{R}$ such that G_{ρ_0} satisfies the property (P), and has the mountain-pass geometry. Then for a dense set of ρ in a neighborhood of ρ_0, there exists $u_\rho \in \mathcal{B} \setminus \{0\}$ such that*

$$G'_\rho(u_\rho) = 0$$

[1] Using a technique used by M. Struwe in [10] page 60, one should be able to improve the statement in lemma 2.2 to almost every ρ in a neighborhood of ρ_0. I. Schindler attempted to prove this in [5], but the proof is not complete.

3 Proof of the main results

The critical sequence converging to the solution of (P_λ) is given by the following lemma which is proved in [5] when $p = 2$. The proof for general p is similar and will be omitted.

Lemma 3.1 *Under assumption* **H3**, *if $\rho \in I$, then G_ρ has critical sequence $(u_n)_n$ which tends either to a local minimizer of G_ρ, or satisfies equations* (6) *and* (5), *and is of mountain-pass type. If the sequence tends to a local minimizer of G_ρ, then $\exists t_0 \neq 0 : \ \|u_n\|_{\mathcal{B}} \longrightarrow t_0$.*

To prove Theorem 1.1 and Theorem 1.2 it suffices to prove that G_ρ satisfies property (P):

Lemma 3.2 *Assume* **H1** *and* **H3**. *Then G_ρ satisfies property (P) for $\rho > \alpha$.*

Proof: Let $(u_n)_n$ be a sequence satisfying (4) and (5). By (4), the sequence $(u_n)_n$ is bounded, which means that there is a renumbered subsequence converging to a weak limit u. In fact, this convergence is strong. To prove this, it suffices to check that for each $\varepsilon > 0$ there exist $R > 0$ such that:

$$\limsup_{n\to\infty} \int_{\Omega\setminus B_R} \{|\nabla u_n|^p + V(x)|u_n|^p\}dx < \varepsilon.$$

Let R be chosen such that $K \subset B_{R/2}$, and let

$$\eta_R = \begin{cases} 0 \text{ on } B_{R/2} \\ 1 \text{ on } \Omega \setminus B_R \end{cases} \text{ with } 0 \leq \eta_R \leq 1 \ \text{ and } |\nabla \eta_R| \leq \frac{C}{R}.$$

Since $(u_n)_n$ is a bounded sequence of Palais-Smale type, then

$$< G_\rho^{'}(u_n), \eta_R u_n >= o(1)$$

i.e:

$$\int_\Omega \{|\nabla u_n|^p + V(x)|u_n|^p\} \eta_R dx + \int_\Omega u_n |\nabla u_n|^{p-2} \nabla u_n . \nabla \eta_R dx = \int_\Omega \lambda f(x, u_n) u_n \eta_R dx + o(1)$$

Using **H1** we obtain

$$\begin{aligned} \int_\Omega \{|\nabla u_n|^p + V(x)|u_n|^p\}\eta_R dx + \int_\Omega u_n |\nabla u_n|^{p-2}\nabla u_n.\nabla\eta_R dx \ &\leq \ \lambda\alpha \int_{\Omega\setminus B_{R/2}} V(x)|u_n|^p \eta_R dx \\ &\quad +o(1) \\ &\leq \ \lambda\alpha \int_\Omega \{|\nabla u_n|^p + V(x) \\ &\quad \cdot |u_n|^p\}\eta_R dx + o(1). \end{aligned}$$

Since $\rho = \dfrac{1}{\lambda}$, we have

$$\begin{aligned}(\rho-\alpha)\int_{\Omega}\{|\nabla u_n|^p+V(x)|u_n|^p\}\eta_R dx &\leq -\rho\int_{\Omega}u_n|\nabla u_n|^{p-2}\nabla u_n.\nabla\eta_R dx+o(1)\\ &\leq \left|\rho\int_{\Omega}u_n|\nabla u_n|^{p-2}\nabla u_n.\nabla\eta_R dx\right|+o(1)\\ &\leq \rho\frac{C}{R}\int_{\Omega}|u_n||\nabla u_n|^{p-1}dx+o(1)\\ &\leq \rho\frac{C}{R}\left(\|u_n\|_{L^p(\Omega)}\|\nabla u_n\|^{\frac{p}{q}}_{L^p(\Omega)}\right)+o(1),\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Thus:

$$\int_{\Omega\setminus B_R}\{|\nabla u_n|^p+V(x)|u_n|^p\}dx \leq \frac{\rho C}{R(\rho-\alpha)}\left(\|u_n\|_{L^p(\Omega)}\|\nabla u_n\|^{\frac{p}{q}}_{L^p(\Omega)}\right)+o(1)$$

which completes the proof.

Lemma 3.3 *If* **H2**–**H5** *hold, then G_ρ satisfies the property (P) for all $\rho\in\mathbb{R}$.*

For the proof, we use the concentration-compactness method. We recall the concentration-compactness lemma used in [5]:

Lemma 3.4 *Let $(\phi_n)_n$ be a sequence of non negatives functions in $L^1(\Omega)$ such that $\|\phi_n\|_{L^1(\Omega)}=t$. Then either:*

1. $\forall R<\infty,\ \lim\limits_{n\to\infty}\sup\limits_{y\in\mathbb{R}^N}\int_{B_R+y}\phi_n(x)dx=0,$
   or

2. *there exist a renumbered subsequence $(\phi_n)_n$, a $\beta\in(0,t]$, and sequences of nonnegative functions $(r_n)_n$, $(\phi_n^1)_n$, $(\phi_n^2)_n$ in $L^1(\mathbb{R}^N)$ such that $(\phi_n^1)_n$ is tight up to translations, and*

$$\phi_n=\phi_n^1+\phi_n^2+r_n,$$

$$\|\phi_n-(\phi_n^1+\phi_n^2)\|_{L^1(\Omega)}=\|r_n\|_{L^1(\Omega)}\xrightarrow[n\to\infty]{}0,$$

$$\|\phi_n^1\|_{L^1(\Omega)}\xrightarrow[n\to\infty]{}\beta,\ \|\phi_n^2\|_{L^1(\Omega)}\xrightarrow[n\to\infty]{}t-\beta,$$

$$dist\left(Supp(\phi_n^1),\ Supp(\phi_n^2)\right)\xrightarrow[n\to\infty]{}\infty,$$

$$\phi_n^i r_n=0\ a.e,\ i=1,2.$$

Furthermore, if there exists $R<\infty$ such that $\lim\limits_{n\to\infty}\int_{B_R}\phi_n dx=\delta>0$, then we may assume the ϕ_n^1 to be tight.

Using the preceding lemma one can prove the following smooth version of concentration compactness:

Lemma 3.5 *Let (u_n) be a sequence in S_t, and let $\phi_n := |\nabla u_n|^p + V(x)|u_n|^p$. Assume $(\phi_n)_n$ is as in lemma 3.4 case 2. Then there exist two sequences $(u_n^1)_n$, $(u_n^2)_n$ in $\mathcal{B}$ such that*

$$\|u_n - (u_n^1 + u_n^2)\|_{\mathcal{B}} \xrightarrow[n\to\infty]{} 0,$$

$$dist\left(Supp(u_n^1), Supp(u_n^2)\right) \xrightarrow[n\to\infty]{} \infty,$$

$$\|u_n^1\|_{\mathcal{B}}^p = \beta,\ \|u_n^2\|_{\mathcal{B}}^p = t - \beta.$$

Moreover, $\psi_n^1 := |\nabla u_n^1|^p + V(x)|u_n^1|^p$ is tight up to translations.

We also recall

Lemma 3.6 *Let $(u_n)_n$ be a sequence in S_t, and assume that* **H2** *and* **H3** *hold.* *If $\displaystyle\int_\Omega |u_n(x)|^p dx \xrightarrow[n\to\infty]{} 0$, then*

$$(g'(u_n), u_n) \xrightarrow[n\to\infty]{} 0.$$

We need also the following lemma originally proved for polynomial V by C. Fefferman and D. H. Phong in [2], and extended to $V \in A_q$ with $1 < q < \infty$ and $p = 2$ by Z. Shen in [7, 8].

Lemma 3.7 *Let $u \in C_c^1(\mathbb{R}^N)$. Then:*

$$\int_{\mathbb{R}^N} |u(x)|^p m(x,V)^p dx \le C \int_{\mathbb{R}^N} |\nabla u(x)|^p + V(x)|u(x)|^p dx$$

To prove this result, we need the following lemma which is the first statement of Lemma 1.8 in [8]. Its proof is based on the fact that $V \in A_p$.

Lemma 3.8 *For $x, y \in \mathbb{R}^N$, $m(x,V) \sim m(y,V)$ if $|x-y| \le \dfrac{1}{m(x,V)}$.*

The proof of Lemma 3.7 is similar to that for $p = 2$ in [7, 8]. It is included for completeness without the confusing typographical error in [7, 8].

Proof of Lemma 3.7: Since $V \in A_p$, then $V \in A_\infty$. Hence, according to (2), there exists $\varepsilon \in]0,1[$ such that for each ball $B \subset \mathbb{R}^N$, we have

$$\left|\left\{y \in B :\ V(y) \ge \frac{\varepsilon}{|B|}\int_B V(x)dx\right\}\right| \ge \frac{1}{2}|B|.$$

Let $x_0 \in \mathbb{R}^N$, $r_0 = \dfrac{1}{m(x_0,V)}$, $B = B(x_0, r_0)$, and $E := \left\{ y \in B : \ V(y) \geq \dfrac{\varepsilon}{|B|} \displaystyle\int_B V(x)dx \right\}$, where ε is chosen such that $\dfrac{\varepsilon}{w_N} \leq 1$. ($w_N = |B(0,1)|$).

For $y \in E$ we have $V(y) \geq \dfrac{\varepsilon}{w_N r_0^N} \displaystyle\int_B V(x)dx = \dfrac{c_0}{r_0^p}$, where $c_0 = \dfrac{\varepsilon}{w_N}$.

Thus, $\displaystyle\int_B \min\left\{\frac{c_0}{r_0^p}, V(y)\right\} dy \geq \int_E \min\left\{\frac{c_0}{r_0^p}, V(y)\right\} dy \geq C r_0^{N-p}$.

On the other hand we have

$$\int_B |\nabla u(x)|^p dx \geq \frac{C}{r_0^{p+N}} \int_B \int_B |u(x) - u(y)|^p dx dy,$$

and

$$\int_B V(x)|u(x)|^p dx \geq \frac{C}{r_0^N} \int_B \int_B V(x)|u(x)|^p dx dy.$$

Summing those two inequalities, we obtain:

$$\begin{aligned}
\int_B |\nabla u(x)|^p dx + \int_B V(x)|u(x)|^p dx &\geq \frac{C}{r_0^N} \int_B \int_B \left\{ \frac{c_0}{r_0^p}|u(x) - u(y)|^p + V(y)|u(y)|^p \right\} dx dy \\
&\geq \frac{C}{r_0^N} \int_B \int_B \min\left\{\frac{c_0}{r_0^p}, V(y)\right\} \left(|u(x) - u(y)|^p + |u(y)|^p\right) dx dy \\
&\geq \frac{C}{r_0^N} \int_B \int_B \min\left\{\frac{c_0}{r_0^p}, V(y)\right\} |u(x)|^p dx dy \\
&\geq C \int_B \frac{1}{r_0^p} |u(x)|^p dx
\end{aligned}$$

Using Lemma 3.8 we obtain:

$$\int_B |u(x)|^p m(x,V)^p dx \leq C \int_B \left\{ |\nabla u(x)|^p + V(x)|u(x)|^p \right\} dx \,.$$

Multiplying this last inequality by $m(x_0,V)^N$ and using Lemma 3.8 we obtain:

$$\begin{aligned}
&\int_{\mathbb{R}^N} |u(x)|^p m(x,V)^{p+N} \chi_{B(x_0,r_0)}(x) dx \\
&\qquad \leq C \int_{\mathbb{R}^N} \left\{ |\nabla u(x)|^p + V(x)|u(x)|^p \right\} m(x,V)^N \chi_{B(x_0,r_0)}(x) dx \,.
\end{aligned}$$

Remarking that $\chi_{B(x_0,r_0)}(x) = \chi_{B(x,r_0)}(x_0)$, and integrating in x_0, we obtain:

$$\begin{aligned}
\int_{\mathbb{R}^N} |u(x)|^p m(x,V)^{p+N} \left(\int_{B(x,r_0)} dx_0 \right) dx \ &\leq \ C \left\{ \int_{\mathbb{R}^N} |\nabla u(x)|^p \, m(x,V)^N \left(\int_{B(x,r_0)} dx_0 \right) dx \right. \\
&\quad + \left. \int_{\mathbb{R}^N} V(x)|u(x)|^p m(x,V)^N \left(\int_{B(x,r_0)} dx_0 \right) dx \right\} .
\end{aligned}$$

With Lemma 3.8, we deduce that

$$\int_{|x_0-x|<r_0} dx_0 \sim \left(\frac{1}{m(x,V)}\right)^N$$

which completes the proof.

Proof of Lemma 3.3: Let $(u_n)_n$ be a sequence satisfying (4) and (5). We may assume that $\|u_n\|_{\mathcal{B}}^p = t$ for every n. Let $\phi_n := |\nabla u_n|^p + V(x)|u_n|^p$ which we claim to be in the case 2 of Lemma 3.4. Suppose this were not the case, then there is a relabeled subsequence $(u_n)_n$ such that

$$\int_{B_R} |\nabla u_n(x)|^p + V(x)|u_n(x)|^p dx \xrightarrow[n\to\infty]{} 0 \ \forall R < \infty \tag{7}$$

According to lemma 3.7, there exists $M > 0$ such that $\displaystyle\int_{\mathbb{R}^N} m(x,V)^p |u_n(x)|^p dx \leq M$.

Since $\displaystyle\lim_{|x|\to\infty} m(x,V) = +\infty$, then $\forall \varepsilon > 0, \exists R > 0$ such that $m(x,V)^p \geq \dfrac{M}{\varepsilon}$ on $\{x: \ |x| \geq R\}$. Hence

$$\int_{\{|x|\geq R\}} |u_n(x)|^p dx \leq \varepsilon \ \ \forall k. \tag{8}$$

From (7) and (8), we conclude that

$$\int_{\Omega} |u_n(x)|^p dx \xrightarrow[n\to\infty]{} 0. \tag{9}$$

According to lemma 3.6 we have:

$$(g'(u_n), u_n) \xrightarrow[n\to\infty]{} 0.$$

Furthermore, (5) and the fact that $(u_n)_n$ is bounded implies that

$$\rho\|u_n\|_{\mathcal{B}}^p - (g'(u_n), u_n) \xrightarrow[n\to\infty]{} 0.$$

Thus $\|u_n\|_{\mathcal{B}}^p = t = 0$, a contradiction. This completes the proof, see [5], [6].

Proof of Theorem 1.1 : By Lemma 3.1, Remark 2.1, and Lemma 2.2, we deduce the first statement. The second one is a consequence of Lemma 3.1 and the fact that G_ρ satisfies property (P).

The proof of Theorem 1.2 is similar.

Acknowledgment: I acknowledge Ian Schindler for useful criticisms and conversations on some questions treated in this paper.

References

[1] **Del Pino, M.**, and **Felmer, P. :** *Local mountain passes for semilinear elliptic problems in unbounded domains.* Calc. Var. Partial Differ. Equ. **4**, 121-137 (1996)

[2] **Fefferman,C. :** *The uncertainty principle.* Bull. Amer. Math. Soc. **9 nª2**, 129-206 (1983)

[3] **Kurata, K. :** *Existence and semi-classical limit of the least energy solution to a nonlinear Schrödinger equation with electromagnetic fields.* Preprint

[4] **Rabinowitz, P. H. :** *On a class of nonlinear Schrödinger equations.* Z. angew. Math. Phys. **43**, 270-291 (1992)

[5] **Schindler, I. :** *A nonlinear Schrödinger equation in $\mathbb{R}^N$.* Reaction Diffusion Systems **194**, Trieste (1997)

[6] **Schindler, I. :** *Quasilinear elliptic boundary value problem on unbounded cylinder and a related montain-pass lemma.* Arch. Rat. Mec. Anal. **120**, 363-374 (1992)

[7] **Shen, Z. :** *L^p Estimation for Schrödinger operators with certain potentials.* Ann. Inst. Fourier, Grenoble **45, 2**, 513-546 (1995)

[8] **Shen, Z. :** *On the Neumann problem for Schrödinger operators in Lipschitz domains.* Indiana Univ. Math. J. **43** (1), 143-176 (1994)

[9] **Stein, E. M. :** *Harmonic Analysis: Real-Variable Method, Orthogonality, and Oscillatory Integrals.* Princeton Univ. Press 1993

[10] **Struwe, M. :** *The existence of surfaces of constant mean curvature with free boundaries.* Acta Mathematica **160**, 19-64 (1988)

[11] **Tintarev, K. :** *Mountain pass and impasse for non-smooth functionals with counstraints.* Nonlinear Analysis, Theory, Methods, and Applications **26**, 798-803 (1996)

received: April 25, 2001

Author:

Nasreddine Megrez
CEREMATH-MIP/UT1 University of Toulouse1
21, allee de Brienne, Bat C, 31000 Toulouse
France

e-mail: megrez@math1.univ-tlse1.fr