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On the Representation of Continuous Solutions of Two-Scale Difference Equations at Dyadic Points

ABSTRACT. The paper gives some insight into the structure of continuous solutions of two-scale difference equations at dyadic points. An example is given in which the solution is estimated.

KEY WORDS. Two-scale difference equations, 2^l-slanted matrices, recursions

Let φ be a continuous compactly supported solution of the two-scaled difference equation (cf. [3])

$$\varphi\left(\frac{t}{2}\right) = \sum_{n=0}^{N} c_n \varphi(t-n) \tag{1}$$

($t \in \mathbb{R}$) with $N \in \mathbb{N}$ (in fact it must be $N \geq 2$), $c_n \in \mathbb{C}$, $c_0 c_N \neq 0$ and

$$\sum_{n=0}^{N} c_n = 2^M ,$$

($M \in \mathbb{N}$). In [2, Corollary 2.5] it was shown that the restriction of φ to $[0,1]$ possesses at dyadic points the representation

$$\varphi\left(\frac{k}{2^l}\right) = c_0^l \sum_{j=1}^{N-1} y_{N+k-j} \varphi(j) \tag{2}$$

($k, l \in \mathbb{N}_0$, $0 \leq k \leq 2^l$) where the coefficients are defined by the initial values

$$y_1 = \cdots = y_{N-1} = 0, \quad y_N = 1 \tag{3}$$

and the recursions

$$c_0 y_k = \sum_{j=\lceil \frac{k}{2} \rceil}^{\lfloor \frac{N+k}{2} \rfloor} c_{N+k-2j} y_j \,. \tag{4}$$

Here $\lfloor . \rfloor$ denotes the floor, and $\lceil . \rceil$ the ceiling function, cf. [4, p. 52]. It is suitable to use the extensions $y_j = 0$ for $j < 0$ and $c_n = 0$ for both $n < 0$ and $n > N$, respectively, and to introduce the infinite two-scale matrix

$$\mathbf{A} = (c_{2j-k}) \qquad (1 \le j, k)\,.$$

Then, for $l \in \mathbb{N}$, the matrix $\mathbf{A}^l$ possesses the entries

$$c_0^l y_{2^l+N-1},\ c_0^l y_{2^l+N-2},\ c_0^l y_{2^l+N-3},\ \ldots \tag{5}$$

in its first row, cf. [2, Theorem 2.4]. It can easily be seen that $\mathbf{A}^l$ is a 2^l-slanted matrix, i.e.

$$\mathbf{A}^l = \left(c^{(l)}_{2^l j-k}\right) \qquad (1 \le j, k) \tag{6}$$

where $c_n^{(1)} = c_n$ and

$$c^{(l+m)}_{2^{l+m}j-k} = \sum_{i=1}^{\infty} c^{(l)}_{2^l j-i} c^{(m)}_{2^m i-k}\,,$$

in particular $c_0^{(l)} = c_0^l$, $c^{(l)}_{(2^l-1)N} = c_N^l$, and $c_n^{(l)} = 0$ for both $n < 0$ and $n > (2^l - 1)N$, respectively.

For our next considerations we need the following submatrices of $\mathbf{A}$:

$$A_l = (c_{2j-k}) \qquad (1 \le j, k \le 2^l + N - 1)$$

with $l \in \mathbb{N}_0$. If A_0 is diagonalizable then there exist matrices $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$ and E with

$$A_0 = E^{-1}\Lambda E\,, \tag{7}$$

where the j-th row $(e_{j1}, \ldots, e_{jN})$ of E is a left eigenvector of A_0 to the eigenvalue $\lambda_i (j \in \{1, \ldots, N\})$. This eigenvector can be continued to a left eigenvector $(e_{j1}, e_{j2}, \ldots)$ of $\mathbf{A}$ to the same eigenvalue. The matrix of these eigenvectors we denote by

$$\mathbf{E} = (e_{jk}) \qquad (1 \le j \le N,\ 1 \le k)\,,$$

and we also need the finite submatrices

$$G_l = (e_{jk}) \qquad (1 \le j \le N,\ 1 \le k \le 2^l + N - 1)\,. \tag{8}$$

Theorem *Let A_0 be diagonalizable. Then with the foregoing notations the first $2^l + N - 1$ terms of (5) can be represented as*

$$c_0^l y_{2^l+N-k} = \sum_{i=1}^{N} \lambda_i^l f_{1i} e_{ik} \qquad (k = 1, \ldots, 2^l + N - 1, l \in \mathbb{N}_0) \tag{9}$$

where $(f_{11}, \ldots, f_{1N})$ is the first row of $E^{-1} = (f_{jk})$.

Proof: The right-hand sides of (9) for $k = 1, \ldots, 2^l + N - 1$ are the entries of the first row of the matrix $E^{-1}\Lambda G_l$. We have to show that they coincide with the first $2^l + N - 1$ entries of the first row of $\mathbf{A}^l$. For $l = 0$ this is clear. For $l \geq 1$ the matrices $\mathbf{A}$ and A_l can be splitted into the following block forms

$$\mathbf{A} = \begin{pmatrix} A_l & * \\ O & * \end{pmatrix}, \quad A_l = \begin{pmatrix} A_0 & * \\ O & * \end{pmatrix},$$

where the asterisks indicate suitable submatrices and O suitable zero matrices. Hence,

$$\mathbf{A}^l = \begin{pmatrix} A_l^l & * \\ O & * \end{pmatrix}, \quad A_l^l = \begin{pmatrix} A_0^l & * \\ O & * \end{pmatrix}, \tag{10}$$

where $A_l^l = (c^{(l)}_{2^l j - k})$ $(1 \leq j, k \leq 2^l + N - 1)$ using the notation (6). Since $c^{(l)}_{2^l j - k} = 0$ for $2^l j - k > (2^l - 1)N$, and therefore for both $N + 1 \leq j$ and $1 \leq k \leq 2^l + N - 1$, we have in fact

$$A_l^l = \begin{pmatrix} A_0^l & * \\ O & O \end{pmatrix}. \tag{11}$$

Comparison of the Jordan normal form

$$A_l = E_l^{-1} \begin{pmatrix} \Lambda & O \\ O & J \end{pmatrix} E_l \tag{12}$$

with (7) and (8) shows that the outer factors must have the block forms

$$E_l^{-1} = \begin{pmatrix} E^{-1} & * \\ O & * \end{pmatrix}, \; E_l = \begin{pmatrix} E & * \\ O & * \end{pmatrix} = \begin{pmatrix} G_l \\ * \end{pmatrix}.$$

Comparison of (11) with (12) implies that $J^l = 0$ and therefore

$$A_l^l = \begin{pmatrix} E^{-1} & * \\ O & * \end{pmatrix} \begin{pmatrix} \Lambda^l G_l \\ O \end{pmatrix} = \begin{pmatrix} E^{-1}\Lambda^l G_l \\ O \end{pmatrix}.$$

Now, the assertion follows from (10) $\square$

Remarks 1°. Choosing in (2) $k = 2^l - m$ then by means of (9) with $k = m + j$ we get some insight into the structure of $\varphi\left(1 - \frac{m}{2^l}\right)$, $0 \leq m \leq 2^l$. Though the result can be used for explicit calculations of φ, this is not recommended.

2°. The entries of the eigenvectors $(e_{i1}, e_{i2}, \ldots)$ satisfy analogous recursions as in (4), namely

$$\lambda_i e_{ik} = \sum_{j=\lceil \frac{k}{2} \rceil}^{\lfloor \frac{N+k}{2} \rfloor} c_{2j-k} e_{ij} \,.$$

3°. In the case $N = 2$ formula (9) was already set up (with other notations) in [1, (3.1)].

4°. The case that A_0 is non-diagonalizable can be treated with some more effort, cf. [1, (3.3)] in the case $N = 2$.

5°. In [2, Proposition 2.7] it must be $m_0 = 0$.

6°. The first column of E^{-1} is a right eigenvector of A_0 to the eigenvector 1. This implies $\varphi(j) = f_{j1}$ (up to a constant factor), cf. [2, (2.4)] with $t = 1$.

7°. Formula (9) can be simplified if the entries f_{1k} of the first row of E^{-1} are normlized according to $f_{1k} = 1$ so far as $f_{1k} \neq 0$. But it is also possible that $f_{1k} = 0$ for a fixed k as in the folowing

Example Choosing $c_0 = \frac{1}{4}$, $c_3 = 1$, $c_4 = \frac{3}{4}$ and $c_n = 0$ otherwise, so that $N = 4$, then $\Lambda = \operatorname{diag}\left(1\ \frac{1}{2}\ -\frac{1}{2}\ \frac{3}{4}\right)$ and

$$E^{-1} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 2 & -2 & 3 \\ 1 & -3 & 1 & -9 \\ 0 & 0 & 0 & 5 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{4} & 0 & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} & 0 & \frac{1}{20} \\ 0 & 0 & 0 & \frac{1}{5} \end{pmatrix}.$$

Hence, (9) yields in particular

$$\begin{aligned} y_{2^l+3} &= 2^{l-1}(1 + (-1)^l), \quad y_{2^l+2} = (1 - (-1)^l)2^{l-2}, \\ y_{2^l+1} &= 0, \quad y_{2^l} = \frac{1}{5}\left(3^l - (5 - (-1)^l)2^{l-2}\right) \end{aligned}$$

($l \in \mathbb{N}_0$). Formula (2) with $\varphi(3) = 1$ and $\varphi(j) = 0$ otherwise specializes to

$$\varphi\left(\frac{k}{2^l}\right) = \frac{1}{4^l} y_{k+1} \tag{13}$$

for $0 \leq k \leq 2^l$. But (13) is even valid for $0 \leq k \leq 3 \cdot 2^l$, since $\varphi(t) = 0$ for $t \leq 0$ and (1) imply $\varphi(\frac{t}{2}) = \frac{1}{4}\varphi(t)$ for $0 \leq t \leq 3$. The recursions (4) specialize to

$$y_{2j} = 3y_j + y_{j+2}, \ y_{2j-1} = 4y_j \tag{14}$$

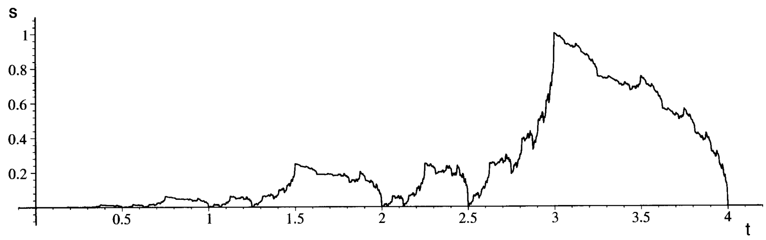
for $j \in \mathbb{N}$, and with the initial values (3) with $N = 4$ we obtain for the first values

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
y_j	0	0	0	1	0	0	4	3	0	4	0	3	16	12	12	13	0	0	16	15	0	16	12	21	64	60

where it easily follows by induction that

$$y_{(2^{2n}+1)2^m+1} = 0 \tag{15}$$

for all $m, n \in \mathbb{N}_0$. The solution $s = \varphi(t)$ is plotted in the following picture:



Introducing the notations

$$x_n = y_{3\cdot 2^n+1},\ z_n = y_{3\cdot 2^n+3},\ u_n = y_{3\cdot 2^n+2},\ v_n = y_{3\cdot 2^n},\ w_n = y_{3\cdot 2^n-1}$$

$(n \in \mathbb{N}_0)$ and using (14) we find the recursions

$$x_n = 4x_{n-1},\ z_n = 12x_{n-2} + 4z_{n-2},$$
$$u_n = 3x_{n-1} + z_{n-1},\ v_n = 3v_{n-1} + u_{n-1},\ w_n = 12v_{n-2} + 4u_{n-2},$$

and by means of the initial values from the forgoing table their solutions

$$x_n = 4^n,\ z_n = 4^n + ((-1)^n - 3)2^{n-1},\ u_n = 4^n - ((-1)^n + 3)2^{n-2},$$
$$v_n = 4^n + 3\cdot 2^{n-2} + \frac{1}{5}((-2)^{n-2} - 3^{n+2}),\ w_n = 4^n + 3\cdot 2^{n-1} + \frac{1}{5}((-2)^{n-1} - 4\cdot 3^{n+1})\,.$$

Proposition *The solutions y_k of (14) with (3) for $N = 4$ satisfy the estimates*

$$0 \le y_k \le \left(\frac{k-1}{3}\right)^2 \tag{16}$$

$(k \in \mathbb{N})$ *where both bounds are sharp for infinitely many k.*

Proof: The first inequality of (16) follows from (14) and the initial values (3) with $N = 4$, the sharpness from (15). For $k = 3\cdot 2^n + 1$ $(n \in \mathbb{N}_0)$ the second inequality is in fact an equality in view of $x_n = 4^n$. For $1 \le k \le 3$ it is trivial. For $k \ne 3\cdot 2^n + 1$ and $k \ge 5$ we shall prove the better inequality

$$y_k \le \frac{1}{9}k(k-2)\,. \tag{17}$$

For $k = 3\cdot 2^n + 2$ $(n \in \mathbb{N}_0)$ we have

$$y_k = u_n \le 4^n - 2^{n-1} = \frac{1}{9}(k-2)\left(k - \frac{7}{2}\right)$$

and (17) is valid. For $k = 3\cdot 2^n - 2$ we have

$$y_k = 3w_{n-1} + x_{n-1} \le 4^n + \frac{12}{5}(2^n - 3^n)$$

and (17) is valid when $n \geq 2$ ($n = 1$ corresponds to $y_4 = 1$).

In order to complete the proof we introduce the sets $M_n = \{3 \cdot 2^n + 2, \ldots, 3 \cdot 2^{n+1}\}$ ($n \in \mathbb{N}_0$). The inequality (17) is valid for $k \in M_0 = \{5, 6\}$. If (17) is valid for $k \in M_n$ then by means of the recursions (14) it follows that (17) is valid for the odd k from M_{n+1}. Analogously, we see that (17) is also valid for the even k from M_{n+1} if we simultaneously take into account the already treated two special cases. Hence by induction, (17) is valid for all $k \in \bigcup_{n=0}^{\infty} M_n$ □

In view of (13) and the continuity of φ we immediately get the

Corollary *For $0 \leq t \leq 3$ the solution of our example for (1) with $\varphi(3) = 1$ satisfies the estimates*

$$0 \leq \varphi(t) \leq \frac{1}{9}t^2$$

where both bound are sharp for infinitely many t.

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