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Fouzi Zaouch

# Global Existence and Boundedness of Solutions of the Time-Dependent Ginzburg-Landau Equations with a Time-Dependent Magnetic Field

ABSTRACT. This paper is concerned with existence, uniqueness and long-time asymptotic behavior of the solutions of the time-dependent Ginzburg-Landau equations of superconductivity, in the case where the applied magnetic field $\mathbf{H}$ is time-dependent. We first prove existence and uniqueness of solutions with $H^1$-initial data. This result is obtained under the "$\phi = -\omega(\nabla \cdot \mathbf{A})$" gauge with $\omega > 0$. These solutions become then uniformly bounded in time for the $H^1$-norm, by assuming time-uniform boundedness on $\mathbf{H}$ and its time derivative.

KEY WORDS. Superconductivity, Ginzburg-Landau equation, gauge, initial boundary value problems, global existence and uniqueness.

## 1 Introduction

In this paper we consider the Ginzburg-Landau model for superconductivity in the nonstationary case. Based on an averaging method of the BCS theory, a time-dependent Ginzburg-Landau model was derived by Gor'kov and Eliashberg in 1968 [1]. The study of this model for superconductivity may give a better understanding of the physical state of a superconductor, especially for the high-temperature superconductors. It is known from the physics literature that the realization of this physical phenomena and then the validation of this model is only possible under temperatures near the critical temperature. The equations describing the state of a superconducting material near the critical temperature are nonlinear differential equations for the *order-parameter* $\psi$, the *vector potential* $\mathbf{A}$ and the *electric potential* $\phi$, whose evolution in presence of a *magnetic field* $\mathbf{H}$ is governed by the following system

$$\eta\left(\frac{\partial}{\partial t} + i\kappa\phi\right)\psi = -\left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)^2\psi + \left(1 - |\psi|^2\right)\psi \quad \text{in} \quad \Omega \times (0, \infty), \tag{1.1}$$

$$\frac{\partial \mathbf{A}}{\partial t} + \nabla\phi = -\nabla \times \nabla \times \mathbf{A} + \mathbf{J}_s + \nabla \times \mathbf{H} \quad \text{in} \quad \Omega \times (0, \infty), \tag{1.2}$$

where $\mathbf{J}_s$ is given by

$$\mathbf{J}_s \equiv \mathbf{J}_s(\psi, \mathbf{A}) = \frac{1}{2i\kappa}(\psi^* \nabla \psi - \psi \nabla \psi^*) - |\psi|^2 \mathbf{A} = -\mathrm{Re}\left[\psi^* \left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)\psi\right]. \tag{1.3}$$

Equations (1.1)-(1.3) are satisfied everywhere in a domain $\Omega$, which is the region occupied by the superconducting material and at all times $t > 0$. The associated boundary conditions are

$$\mathbf{n} \cdot \left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)\psi + \frac{i}{\kappa}\gamma\psi = 0 \quad \text{and} \quad \mathbf{n} \times (\nabla \times \mathbf{A} - \mathbf{H}) = \mathbf{0} \quad \text{on} \quad \partial\Omega, \tag{1.4}$$

where $\partial\Omega$ is the boundary of $\Omega$ and $\mathbf{n}$ the local outer unit normal to $\partial\Omega$. They must be satisfied at all times $t > 0$. Henceforth, the term „TDGL Equations“ refers to the system of equations (1.1)-(1.4).

We assume that $\Omega$ is a bounded domain in $\mathbb{R}^n$ ($n = 2$ or 3) with a boundary $\partial\Omega$ of class $C^{1,1}$. The parameters appearing in the TDGL equations are dimensionless physical constants; $\eta$ is the friction coefficient and $\kappa$ is the Ginzburg-Landau parameter. Here $\eta$ measures the temporal rate of change and the value of $\kappa$ determines the type of superconductor: $\kappa \leq 1/\sqrt{2}$ describes what is known as a *type I* superconductor and $\kappa \geq 1/\sqrt{2}$ as a *type II*. The function $\gamma$ is defined and Lipschitz continuous on $\partial\Omega$ and $\gamma(x) \geq 0$ for $x \in \partial\Omega$. We use the following common notation: $\nabla \equiv$ grad, $\nabla\cdot \equiv$ div, $\nabla\times \equiv$ curl and $\nabla^2 = \nabla \cdot \nabla \equiv \Delta$, $i$ is the imaginary unit and a superscript$^*$ denotes the complex conjugation.

The order parameter $\psi$ is a complex-valued function, it describes the center-of-mass motion of the "superelectron", whose density is $n_s = |\psi|^2$ and whose flux is $\mathbf{J}_s$. $\psi = 0$ corresponds to the normal state, and in a perfect superconducting state $|\psi| = 1$. The vector potential $\mathbf{A}$ takes its values in $\mathbb{R}^n$, it represents the *magnetic potential*, i.e. $\mathbf{B} = \nabla \times \mathbf{A}$. The scalar potential $\phi$ determines the *electric field* $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi$. The vector $\mathbf{H}$ represents the (externally) *applied magnetic field*; it is a given function of space and time, which is divergence free, $\nabla \cdot \mathbf{H} = 0$ at all time. The difference $\mathbf{M} = \mathbf{B} - \mathbf{H}$ is known as the *magnetisation*. The trivial solution ($\psi = 0,\ \mathbf{B} = \mathbf{H},\ \mathbf{E} = 0$) represents the normal state, where all superconducting properties have been lost. For further physics details about Ginzburg-Landau equations, one may consult [1] or [2].

Several works have been devoted recently to questions of existence, uniqueness and long time asymptotic behavior of the solutions of equations (1.1)-(1.4) when the applied magnetic field is stationary, i.e. $\mathbf{H}(t) = \mathbf{H}_0$; as a bibliographical review, we refer to [3], [4], [5], [6], [7] and [8]. To overcome the uniqueness deficiency in equations (1.1)-(1.4), the authors in the mentioned references adopted some *gauge* transformation like the zero-electric gauge ($\phi = 0$), the London gauge ($\nabla \cdot \mathbf{A} = 0$) or the Lorentz gauge ($\phi = -\nabla \cdot \mathbf{A}$). On the other hand, it is known that in presence of an applied time-independent magnetic field $\mathbf{H}$, the TDGL equations enjoy the free energy functional, whose advantage is the getting of some estimates on the solutions.

In contrast to the above situation, Fleckinger, Kaper and Takáč considered in [9] equations (1.1)-(1.4) with a time-dependent magnetic field $\mathbf{H}(t)$. They established in the general context of "$\phi = -\omega(\nabla \cdot \mathbf{A})$" gauge ($\omega > 0$) the existence of a dynamical process. However, some regularities of the solutions obtained are lost in the limit case $\omega = 0$. When $\mathbf{H}$ is stationary, this process becomes a dynamical system enjoying the existence of a global attractor. Subsequently Kaper and Takáč [11] proved that in the special case where the applied magnetic field is asymptotically stationary, the dynamical process generated by the TDGL equations is asymptotically autonomous, i.e. its large-time asymptotic limit is a dynamical system, whose attractor coincides with the one of the dynamical process.

In this paper, we present new, more general results concerning existence, uniqueness and regularity of solutions to the TDGL equations when the applied magnetic field $\mathbf{H}$ exhibits strong temporal fluctuations. In practice $\mathbf{H}$ is either time-independent or time-periodic. For instance, we are able to show global existence for all times $t \geq 0$ if $\mathbf{H}$ is time-periodic. The Lyapunov functional method applied in [9], [10] and [11] is not suitable for treating other than weak temporal fluctuations that disappear for large time with certain convergence rate. Our method of proving global existence and boundedness of solutions for all times $t \geq 0$ significantly improves and extends the classical Lyapunov functional method. Our discussion will rely on the choice of the "$\phi = -\omega(\nabla \cdot \mathbf{A})$" gauge ($\omega > 0$), introduced in [10]. We omit the degenerate case $\omega = 0$. The outline of the paper is as follows. In section 2, we introduce preliminary material, gauge invariance among others, and recall basic results for use in subsequent sections. In section3, we first homogenize the boundary conditions, give definitions of the function spaces we are going to use and assumptions on the data, and after we reformulate the problem into an equivalent abstract initial value problem. Section 4 contains results concerning existence, uniqueness and regularity of solutions to the original equations, the proof of local existence is based on the contraction mapping principle, while global existence is derived from estimates on the energy type functional. In our existence result, we obtain solutions of the TDGL equations from $H^1$-initial data and without requiring $L^\infty$-bound of the initial order parameter $\psi_0$. In section 5, we establish that the solutions obtained become uniformly bounded with respect to $t \geq 0$, this will lead to the existence of an absorbing set for the process.

## 2 Preliminaries

The TDGL equations are not mathematically well posed unless some gauge fixing has been done. It is known in [12] that the solutions of equations (1.1)-(1.4) are unique up to a gauge transformation $\mathcal{G}_\chi$

$$\mathcal{G}_\chi : (\psi,\ \mathbf{A},\ \phi) \longrightarrow \left(\psi e^{i\kappa\chi},\ \mathbf{A} + \nabla\chi,\ \phi - \frac{\partial\chi}{\partial t}\right),$$

here $\chi$ is a given real-valued function (sufficiently smooth) of position and time. In our investigation we adopt the "$\phi = -\omega(\nabla \cdot \mathbf{A})$" gauge. We restrict ourselves to the case $\omega > 0$. Formaly we determine this gauge by taking $\chi \equiv \chi_\omega(x,t)$ as a solution of the following boundary value problem

$$\begin{aligned} \frac{\partial\chi}{\partial t} - \omega\Delta\chi &= \phi + \omega(\nabla \cdot \mathbf{A}) \quad \text{in} \quad \Omega \times (0,\infty), \\ \mathbf{n} \cdot (\nabla\chi) &= -\mathbf{n} \cdot \mathbf{A} \quad \text{on} \quad \partial\Omega \times (0,\infty). \end{aligned}$$

The initial condition $\chi(\cdot,0) = \chi_0$ can be chosen arbitrarily. By virtue of the current gauge, $\mathbf{A}$ and $\phi$ satisfy the identities

$$\phi + \omega(\nabla \cdot \mathbf{A}) = 0 \quad \text{in} \quad \Omega \times (0,\infty), \tag{2.1}$$

$$\mathbf{n} \cdot \mathbf{A} = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty). \tag{2.2}$$

On the other hand the TDGL equations may be given as

$$\eta\frac{\partial\psi}{\partial t} = -\left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)^2 \psi + i\eta\kappa\omega\psi(\nabla \cdot \mathbf{A}) + \left(1 - |\psi|^2\right)\psi \text{ in } \Omega \times (0,\infty), \tag{2.3}$$

$$\frac{\partial\mathbf{A}}{\partial t} = -\nabla \times \nabla \times \mathbf{A} + \omega\nabla(\nabla \cdot \mathbf{A}) + \mathbf{J}_s + \nabla \times \mathbf{H} \quad \text{in} \quad \Omega \times (0,\infty), \tag{2.4}$$

where $\mathbf{J}_s$ is given by (1.3) and the boundary conditions become

$$\mathbf{n} \cdot \nabla\psi + \gamma\psi = 0,\ \mathbf{n} \cdot \mathbf{A} = 0,\ \mathbf{n} \times (\nabla \times \mathbf{A} - \mathbf{H}) = \mathbf{0} \text{ on } \partial\Omega \times (0,\infty). \tag{2.5}$$

For the initial condition, we put

$$\psi(\cdot,0) = \psi_0 \quad \text{and} \quad \mathbf{A}(\cdot,0) = \mathbf{A}_0 \ \text{ in } \ \Omega, \tag{2.6}$$

where $\psi_0$ and $\mathbf{A}_0$ are given.

Now we introduce notations conventions concerning functional spaces, in order to reformulate the gauged TDGL equations (2.3)-(2.6) as an abstract evolution equation in a real Banach space. Throughout, for $p \geq 1$, $L^p(\Omega)$ will denote the usual Lebesgue space, with the norm $\|\cdot\|_p$ , $(\cdot,\cdot)$ is the usual inner-product in $L^2(\Omega)$. For nonnegative integer $m$, we will denote by $H^m(\Omega)$ the usual Sobolev space, with norm $\|\cdot\|_{H^m}$. In the case of nonintegers $m$, $H^m(\Omega)$ is the fractional Sobolev space defined by interpolation, see [12]. The corresponding spaces of complex-valued functions will be denoted by $\mathcal{L}^p(\Omega)$ and $\mathcal{H}^m(\Omega)$ and the corresponding spaces of vector valued functions will be denoted by $\mathbf{L}^p(\Omega)$ and $\mathbf{H}^m(\Omega)$. Without any possible ambiguity, we use the same symbol $\|\cdot\|_p$ to indicate the norms in $\mathcal{L}^p(\Omega)$ and $\mathbf{L}^p(\Omega)$, and the inner-product for $p=2$ is defined in the usual way. We sometimes use $\|\cdot\|_X$ to denote the norm defined on a Banach space $X$. To fix the time-dependence of the functions entering equations (2.3)-(2.5), we define the following spaces: For any given $T>0$, $p\geq 1$ and any given Banach space $X$,

$$L^p(0,T;X)=\left\{u:t\in(0,T)\to u(\cdot,t)\in X \text{ measurable, and } \int_0^T \|u(\cdot,t)\|_X^p\,\mathrm{d}t<\infty\right\},$$

$$L^\infty(0,T;X)=\left\{u:t\in(0,T)\to u(\cdot,t)\in X \text{ measurable, and } \sup_{0<t<T}\|u(\cdot,t)\|_X<\infty\right\},$$

$$W^{1,p}(0,T;X)=\left\{u\in L^p(0,T;X) \text{ absolutely continuous }:\frac{\partial u}{\partial t}\in L^p(0,T;X)\right\}.$$

The spaces $W^{m,p}(0,T;X)$ are defined in similar ways. $C(0,T;X)$ denotes the space of continuously X-valued functions defined in $[0,T]$.

For later purpose we recall some known inequalities and formulas concering vector-valued functions, details and proofs are contained in [13] and [14].

**Poincaré inequality**: For all $\mathbf{A}\in\mathbf{H}^1(\Omega)$, with $\mathbf{n}\cdot\mathbf{A}=0$ on $\partial\Omega$

$$\lambda_0\|\mathbf{A}\|_{\mathbf{H}^1}^2\leq\|\nabla\times\mathbf{A}\|_2^2+\|\nabla\cdot\mathbf{A}\|_2^2, \tag{2.7}$$

$\lambda_0$ is a positive constant.

**Green's formulas**:

(i) For any $\mathbf{A}\in\mathbf{H}(\mathrm{div};\Omega):=\{\mathbf{A}\in\mathbf{L}^2(\Omega):\nabla\cdot\mathbf{A}\in L^2(\Omega)\}$ and $\varphi\in H^1(\Omega)$

$$\int_\Omega(\nabla\cdot\mathbf{A})\varphi\,\mathrm{d}x+\int_\Omega\mathbf{A}\cdot(\nabla\varphi)\,\mathrm{d}x=\int_{\partial\Omega}(\mathbf{n}\cdot\mathbf{A})\varphi\,\mathrm{d}\sigma(x). \tag{2.8}$$

(ii) For any $\mathbf{A}\in\mathbf{H}(\mathrm{curl};\Omega):=\{\mathbf{A}\in\mathbf{L}^2(\Omega):\nabla\times\mathbf{A}\in\mathbf{L}^2(\Omega)\}$ and $\mathbf{B}\in\mathbf{H}^1(\Omega)$

$$\int_\Omega(\nabla\times\mathbf{A})\cdot\mathbf{B}\,\mathrm{d}x-\int_\Omega\mathbf{A}\cdot(\nabla\times\mathbf{B})\,\mathrm{d}x=\int_{\partial\Omega}\mathbf{B}\cdot(\mathbf{A}\times\mathbf{n})\,\mathrm{d}\sigma(x). \tag{2.9}$$

**Gronwall's inequality**: Let $\eta(t)$ be a positive, absolutely continuous function on $[0,T]$, $T>0$, satisfying $\eta'(t)\leq\mu(t)\eta(t)+\nu(t)$ a.e. $t\in[0,T]$, where $\mu$ and $\nu$ are integrable on

$[0,T]$, then

$$\eta(t) \le e^{\int_0^t \mu(s)ds}\left[\eta(0) + \int_0^t e^{-\int_0^s \mu(r)dr}\nu(s)\, ds\right] \quad \text{for all} \quad t \in [0,T]. \tag{2.10}$$

## 3 Abstract Equation

Before we start to reformulate the gauged TDGL equations (2.3)-(2.6) into an equivalent abstract initial-value problem, we turn the boundary condition in the right hand side of (2.5) into a homogenous one. This is achieved at each fixed instant. At each time $t$, assume $\mathbf{H} \in \mathbf{L}^2(\Omega)$ and consider $\mathbf{A_H}$ the unique weak solution of the strongly elliptic boundary-value problem

$$\nabla \cdot \mathbf{A_H} = 0 \qquad \text{and} \qquad \nabla \times \nabla \times \mathbf{A_H} = \nabla \times \mathbf{H} \quad \text{in} \quad \Omega, \tag{3.1}$$

$$\mathbf{n} \cdot \mathbf{A_H} = 0 \qquad \text{and} \qquad \mathbf{n} \times (\nabla \times \mathbf{A_H} - \mathbf{H}) = 0 \quad \text{on} \quad \partial\Omega. \tag{3.2}$$

The existence of $\mathbf{A_H}$ is guaranted by the Lax-Milgram theorem applied to the continuous and coercive bilinear form

$$Q(\mathbf{A},\mathbf{B}) = \int_\Omega (\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{B})\, dx + \omega \int_\Omega (\nabla \cdot \mathbf{A})(\nabla \cdot \mathbf{B})\, dx,$$

on the space $\{\mathbf{A} \in \mathbf{H}^1(\Omega) : \mathbf{n} \cdot \mathbf{A} = 0 \text{ on } \partial\Omega\}$.

The mapping $\mathbf{H} \in \mathbf{L}^2(\Omega) \longmapsto \mathbf{A_H} \in \mathbf{H}^1(\Omega)$ is linear, time independent and continuous, see [9].

The gauged TDGL equations (2.3)-(2.4) are equivalent to a problem in terms of $\psi$ and the reduced vector potential $\tilde{\mathbf{A}} := \mathbf{A} - \mathbf{A_H}$

$$\eta\frac{\partial \psi}{\partial t} = -\left(\frac{i}{\kappa}\nabla + \tilde{\mathbf{A}} + \mathbf{A_H}\right)^2 \psi + i\eta\kappa\omega\psi(\nabla \cdot \tilde{\mathbf{A}}) + \left(1 - |\psi|^2\right)\psi \quad \text{in} \quad \Omega \times (0,\infty), \tag{3.3}$$

$$\frac{\partial \tilde{\mathbf{A}}}{\partial t} = -\nabla \times \nabla \times \tilde{\mathbf{A}} + \omega\nabla(\nabla \cdot \tilde{\mathbf{A}}) + \tilde{\mathbf{J}}_s - |\psi|^2\mathbf{A_H} - \frac{\partial \mathbf{A_H}}{\partial t} \quad \text{in} \quad \Omega \times (0,\infty), \tag{3.4}$$

where $\tilde{\mathbf{J}}_s = \mathbf{J}_s(\psi, \tilde{\mathbf{A}})$ is given by the expression in (1.3), and the boundary condition (2.5) reduces to

$$\mathbf{n} \cdot \nabla\psi + \gamma\psi = 0,\ \mathbf{n} \cdot \tilde{\mathbf{A}} = 0 \quad \text{and} \quad \mathbf{n} \times (\nabla \times \tilde{\mathbf{A}}) = \mathbf{0} \quad \text{on} \quad \partial\Omega \times (0,\infty). \tag{3.5}$$

The supplemented initial condition is

$$\psi(\cdot,0) = \psi_0 \ \text{ and } \ \tilde{\mathbf{A}}(\cdot,0) = \tilde{\mathbf{A}}_0 = \mathbf{A}_0 - \mathbf{A_H}(0) \ \text{ in } \ \Omega. \tag{3.6}$$

We come now to introduce a convenient abstract frame for the system of equations (3.3)-(3.6). In the sequel we will consider the solutions $\psi$ and $\tilde{\mathbf{A}}$ of the system of equations (3.3)-(3.6) as a vector representing the pair

$$\tilde{u} = (\psi, \tilde{\mathbf{A}}) = (\psi, \mathbf{A} - \mathbf{A_H}), \tag{3.7}$$

so we adopt the notations

$$\mathbb{L}^p(\Omega) = \mathcal{L}^p(\Omega) \times \mathbf{L}^p(\Omega) \quad \text{and} \quad \mathbb{H}^s(\Omega) = \mathcal{H}^s(\Omega) \times \mathbf{H}^s(\Omega),$$

and indicate, without any possible confusion, the norm in $\mathbb{L}^p(\Omega)$ by $\|\cdot\|_p$. We set $X = \mathbb{L}^2(\Omega)$ and define some suitable operators related to the dissipative terms in (3.3) and (3.4), we define two linear operators $L_1$ and $L_2$ respectively from $\mathcal{H}^1(\Omega)$ and $\mathbf{H}^1(\Omega)$ to their dual spaces by

$$(L_1\psi, \phi) = \int_\Omega \nabla\psi \cdot \nabla\phi^* \, \mathrm{d}x + \int_{\partial\Omega} \gamma\psi\phi^* \, \mathrm{d}\sigma(x), \tag{3.8}$$

$$(L_2\mathbf{A}, \mathbf{B}) = \int_\Omega (\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{B}) \, \mathrm{d}x + \omega \int_\Omega (\nabla \cdot \mathbf{A})(\nabla \cdot \mathbf{B}) \, \mathrm{d}x. \tag{3.9}$$

Operators $L_1$ and $L_2$ are selfadjoint and positive definite. Moreover the classical theory of second order differential operators allows the extension of $L_1$ and $L_2$ as unbounded linear selfadjoint operators respectively on $\mathcal{L}^2(\Omega)$ and $\mathbf{L}^2(\Omega)$, in which case $L_1\psi = -\Delta\psi$ and $L_2\mathbf{A} = \nabla \times \nabla \times \mathbf{A} - \omega\nabla(\nabla \cdot \mathbf{A})$ in $\Omega$, with

$$\begin{aligned} \mathcal{D}(L_1) &= \{\psi \in \mathcal{H}^2(\Omega) : \mathbf{n} \cdot \nabla\psi + \gamma\psi = 0 \text{ on } \partial\Omega\}, \\ \mathcal{D}(L_2) &= \{\mathbf{A} \in \mathbf{H}^2(\Omega) : \mathbf{n} \cdot \mathbf{A} = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

Let $\mathcal{A}$ be the linear selfadjoint operator in $X$ defined by

$$\begin{aligned} &\mathcal{D}(\mathcal{A}) = \mathcal{D}(L_1) \times \mathcal{D}(L_2), \\ &\mathcal{A}v = \left(-\frac{1}{\eta\kappa^2}\Delta\psi \ , \ \nabla \times \nabla \times \mathbf{A} - \omega\nabla(\nabla \cdot \mathbf{A})\right), \ v = (\psi, \mathbf{A}) \in \mathcal{D}(\mathcal{A}) \, . \end{aligned} \tag{3.10}$$

Since $\mathcal{A}$ is positive definite on $X$, it is then a sectorial operator. It follows that $-\mathcal{A}$ is the infinitesimal generator of an holomorphic semigroup $(\mathrm{e}^{-\mathcal{A}t})_{t\geq 0}$, see [15] and [16], Fractional powers $\mathcal{A}^\alpha$ are well defined for $\alpha \in \mathbb{R}$, they are unbounded for $\alpha > 0$ and $X^\alpha := \mathcal{D}(\mathcal{A}^\alpha)$ is a closed linear subspace of $\mathbb{H}^{2\alpha}(\Omega)$ for $0 < \alpha < 1$ and contains the range of $\mathrm{e}^{-\mathcal{A}t}$ for $\alpha \geq 0$ and $t > 0$. In particular we have

$$X^{1/2} = \{v = (\psi, \mathbf{A}) \in \mathbb{H}^1(\Omega) : \mathbf{n} \cdot \mathbf{A} = 0 \text{ on } \partial\Omega\}. \tag{3.11}$$

In general it is possible to consider $\mathcal{A}$ as a linear operator in $\mathbb{L}^p(\Omega)$ with $1 < p < \infty$, we will use the same symbol $\mathcal{A}$ if no confusion is possible. In this case the $L^p$-theory for elliptic differential operators proves that $-\mathcal{A}$ generates an holomorphic semigroup $(\mathrm{e}^{-\mathcal{A}t})_{t\geq 0}$ in $\mathbb{L}^p(\Omega)$.

On the other hand we consider the initial value problem for the transformed solution $\tilde{u} = (\psi, \tilde{\mathbf{A}})$

$$\frac{\mathrm{d}\tilde{u}}{\mathrm{d}t} + \mathcal{A}\tilde{u} = \mathcal{F}(t, \tilde{u}(t)) \quad \text{for} \quad t > 0 \quad \text{and} \quad \tilde{u}(0) = \tilde{u}_0, \tag{3.12}$$

in $X$, where $\mathcal{F}(t, \tilde{u}) = (\varphi, \mathbf{F})$, $\tilde{u}_0 = (\psi_0, \tilde{\mathbf{A}}_0)$, $\varphi$ and $\mathbf{F}$ are given by the following

$$\begin{aligned} \varphi \equiv \varphi(t, \psi, \tilde{\mathbf{A}}) = \frac{1}{\eta}\Big[ -\frac{2i}{\kappa}(\nabla\psi) \cdot (\tilde{\mathbf{A}} + \mathbf{A}_{\mathbf{H}}) - \frac{i}{\kappa}(1 - \eta\kappa^2\omega)\psi(\nabla \cdot \tilde{\mathbf{A}}) \\ -\psi|\tilde{\mathbf{A}} + \mathbf{A}_{\mathbf{H}}|^2 + \left(1 - |\psi|^2\right)\psi\Big], \end{aligned} \tag{3.13}$$

$$\mathbf{F} \equiv \mathbf{F}(t, \psi, \tilde{\mathbf{A}}) = \tilde{\mathbf{J}}_s - |\psi|^2\mathbf{A}_{\mathbf{H}} - \frac{\partial \mathbf{A}_{\mathbf{H}}}{\partial t}. \tag{3.14}$$

Let $\tilde{u}_0 \in \mathbb{H}^1(\Omega)$, we say that $\tilde{u}$ is a *mild solution* of equation (3.12) on the interval $[0, T]$, for some $T \in (0, \infty)$, if $\tilde{u} : [0, T] \longrightarrow \mathbb{H}^1(\Omega)$ is continuous and

$$\tilde{u}(t) = \mathrm{e}^{-\mathcal{A}t}\tilde{u}_0 + \int_0^t \mathrm{e}^{-\mathcal{A}(t-s)}\mathcal{F}(s, \tilde{u}(s))\,\mathrm{d}s \quad \text{for} \quad 0 \leq t \leq T. \tag{3.15}$$

In particular a mild solution plays the role of a weak solution $(\psi, \tilde{\mathbf{A}})$ for the system of equation (3.3)-(3.5). Of course the existence of a weak solution $u = (\psi, \mathbf{A})$ to the gauged TDGL equations (2.3)-(2.5) requires some regularity about $\mathbf{A}_{\mathbf{H}}$; this suggests that some control should be imposed on the time-dependence of $\mathbf{H}$. Clearly, in definition (3.15) of mild solution, the action of the semigroup $(\mathrm{e}^{-\mathcal{A}t})$ on $\mathcal{F}$ is in $\mathbb{L}^{3/2}(\Omega)$, this is because $\mathcal{F}$ maps $[0, T] \times \mathbb{H}^1(\Omega)$ in $\mathbb{L}^{3/2}(\Omega)$, so it is to distinguish that the operator $\mathcal{A}$ appearing under the symbol integral in (3.15) is considered in $\mathbb{L}^{3/2}(\Omega)$. Furthermore we see that the regularity of the integral in (3.15) introduced by the term $\frac{\partial \mathbf{A}_{\mathbf{H}}}{\partial t}$, namely $\int_0^t \mathrm{e}^{-L_2(t-s)}\frac{\partial \mathbf{A}_{\mathbf{H}}}{\partial s}(s)\,\mathrm{d}s$, determines the regularity of the mild solution $\tilde{u}$ of equation (3.12).

## 4 Existence and Uniqueness

In this section, we study the existence and uniqueness of a mild solution of the initial value problem (3.12). We assume the applied magnetic field $\mathbf{H}(t)$ in $\mathbf{L}^2(\Omega)$ at each $t \geq 0$ and

$(\mathbf{H_0})$ $\quad \mathbf{H} \in L^\infty\left(0, T; \mathbf{L}^2(\Omega)\right) \cap W^{1,2}\left(0, T; \mathbf{L}^2(\Omega)\right), \quad 0 < T < \infty.$

Note that by virtue of [9], $(\mathbf{H}_0)$ implies

$$t \in [0, T] \longrightarrow \int_0^t \mathrm{e}^{-L_2(t-s)}\frac{\partial \mathbf{A}_{\mathbf{H}}}{\partial t}(s)\,\mathrm{d}s \in \mathbf{H}^1(\Omega) \quad \text{is Hölder continuous.} \tag{4.1}$$

**Theorem 1** *For every initial data* $\tilde{u}_0 = (\psi_0, \tilde{\mathbf{A}}_0) \in X^{1/2}$ *the initial value problem* (3.12) *has a unique mild solution* $\tilde{u} = (\psi, \tilde{\mathbf{A}})$ *such that*

$$\tilde{u} \in C\left(0, T; \mathbb{H}^1(\Omega)\right) \cap W^{1,2}\left(0, T; \mathbb{L}^2(\Omega)\right).$$

**Proof:** The proof of local existence and uniqueness is based on the contraction mapping principle. To this goal, we construct a Banach space $C\left(0, \tau; \mathbb{H}^1(\Omega)\right)$ ($\tau$ *small enough*) such that the mapping $\mathcal{G}$ defined from the integral equation in (3.15), namely

$$\mathcal{G}\tilde{u}(t) = \mathrm{e}^{-\mathcal{A}t}\tilde{u}_0 + \int_0^t \mathrm{e}^{-\mathcal{A}(t-s)}\mathcal{F}(s, \tilde{u}(s))\, \mathrm{d}s, \tag{4.2}$$

acts as a contraction map on some closed subset. We need to prove the following properties

$$\mathcal{F}(t, \cdot) : \mathbb{H}^1(\Omega) \longrightarrow \mathbb{L}^{3/2}(\Omega) \text{ is locally Lipschitz for each } t \in [0, T], \tag{4.3}$$

$$\mathrm{e}^{-\mathcal{A}t} : \mathbb{L}^{3/2}(\Omega) \longrightarrow \mathbb{H}^1(\Omega) \text{ for } t > 0 \text{ and } \int_0^\tau \|\mathrm{e}^{-\mathcal{A}t}\|_{\mathcal{L}(\mathbb{L}^{3/2}, \mathbb{H}^1)}\, \mathrm{d}t < \infty. \tag{4.4}$$

Given (4.3) and (4.4), the standard proof of [15, theorem 3.3.3] can be used; we show that there are some positive constants $\tau$ and $\varepsilon$ both small enough such that if we denote $\mathcal{X} = \left\{v \in C\left(0, \tau; X^{1/2}\right) : v(0) = \tilde{u}_0, \|v(t) - \tilde{u}_0\|_{\mathbb{H}^1} \leq \varepsilon\right\}$, then $\mathcal{G} : \mathcal{X} \to \mathcal{X}$ is a contraction map and hence possesses a unique fixed point.

In order to establish (4.3), we need to estimate each term separately. Let two elements $\tilde{u}_1 = (\psi_1, \tilde{\mathbf{A}}_1)$ and $\tilde{u}_2 = (\psi_2, \tilde{\mathbf{A}}_2)$ of $\mathbb{H}^1(\Omega)$, we have for example

$$\begin{aligned} \|\nabla\psi_2 \cdot \tilde{\mathbf{A}}_2 - \nabla\psi_1 \cdot \tilde{\mathbf{A}}_1\|_{3/2} &\leq \|\nabla(\psi_2 - \psi_1)\|_2 \|\tilde{\mathbf{A}}_2\|_6 + \|\nabla\psi_1\|_2 \|\tilde{\mathbf{A}}_2 - \tilde{\mathbf{A}}_1\|_6 \\ &\leq C\|\tilde{u}_2 - \tilde{u}_1\|_{\mathbb{H}^1}, \end{aligned}$$

where $C$ is a positive constant depending only on the norm of $\tilde{u}_1$ and $\tilde{u}_2$ in $\mathbb{H}^1(\Omega)$. Here we have used the continuous Sobolev imbedding of $H^1(\Omega)$ in $L^6(\Omega)$. For the other terms in $\mathcal{F}$, we argue analogously. It follows that if $B_R$ denotes the ball of radius $R$ centered at the origin in $\mathbb{H}^1(\Omega)$, then

$$\|\mathcal{F}(t, \tilde{u}_1) - \mathcal{F}(t, \tilde{u}_2)\|_{3/2} \leq C\|\tilde{u}_1 - \tilde{u}_2\|_{\mathbb{H}^1} \quad \text{for all} \quad \tilde{u}_1, \tilde{u}_2 \in B_R, \tag{4.5}$$

$C$ is the Lipschitz constant, it depends on $R$ but not on $t$.

The proof of the claim in (4.4) uses the smoothing action of the semigroup $\mathrm{e}^{-\mathcal{A}t}$ and some imbedding theorems established for second-order elliptic differential operators. More precisely, rather than (4.4) we can check

$$\|e^{-\mathcal{A}t}\|_{\mathcal{L}(\mathbb{L}^{3/2},\mathbb{H}^1)} \leq Ct^{-\gamma}e^{-\delta t} \quad \text{for all} \quad t>0, \tag{4.6}$$

for some positive constants $C$, $\delta$ and $\gamma > 3/4$ independent on $t$. We refer to [14, theorem 1.6.1] for the proof of this, see also [16].

Next, to show the solution $\tilde{u} = (\psi, \tilde{\mathbf{A}})$ of equation (3.12) is global, some estimates on the energy type functional defined in $\mathbb{H}^1(\Omega)$ by

$$\begin{aligned} E_\omega[\psi,\mathbf{A}] = \int_\Omega \Bigg[ \left|\left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)\psi\right|^2 + \frac{1}{2}\left(1-|\psi|^2\right)^2 + 2\omega(\nabla\cdot\mathbf{A})^2 \\ + |\nabla\times\mathbf{A}-\mathbf{H}|^2\Bigg]\, dx + \frac{1}{\kappa^2}\int_{\partial\Omega}\gamma|\psi|^2\, d\sigma(x), \end{aligned} \tag{4.7}$$

are needed. In fact, from the consideration on $\mathbf{H}$ stated in $(\mathbf{H_0})$, it can be shown ( see [9]), that the pair $u = (\psi, \mathbf{A})$ related to $\tilde{u} = (\psi, \tilde{\mathbf{A}})$ by (3.7) satisfies

$$u \in L^\infty\left(0,T;\mathbb{H}^1(\Omega)\right) \cap W^{1,2}\left(0,T;\mathbb{L}^2(\Omega)\right) \quad \text{and} \quad \nabla\cdot\mathbf{A} \in L^2\left(0,T;\mathbb{H}^1(\Omega)\right).$$

Thus, again by $(\mathbf{H_0})$, we obtain

$$\tilde{u} \in L^\infty\left(0,T;\mathbb{H}^1(\Omega)\right) \cap W^{1,2}\left(0,T;\mathbb{L}^2(\Omega)\right).$$

However, this regularity result concerning $\tilde{u}$ can be improved by the smoothness of the action of $e^{-\mathcal{A}t}$ to prove continuity of $\tilde{u}$. In fact, we have as claimed in (4.1) the map $t \in [0,T] \longrightarrow \int_0^t e^{-L_2(t-s)}\frac{\partial \mathbf{A_H}}{\partial t}(s)\, ds \in \mathbf{H}^1(\Omega)$ is continuous, it suffices then to show that

$$t \in [0,T] \longrightarrow \int_0^t e^{-\mathcal{A}(t-s)}\mathcal{F}'\left(s,\tilde{u}(s)\right)\, ds \in \mathbb{H}^1(\Omega) \quad \text{is continuous,}$$

where $\mathcal{F}'\left(t,\tilde{u}(t)\right) = \mathcal{F}\left(t,\tilde{u}(t)\right) + \left(0, \frac{\partial \mathbf{A_H}}{\partial t}(t)\right)$. At first, we remark that

$$\left(t \longrightarrow \mathcal{F}'(t,\tilde{u}(t))\right) \in L^\infty\left(0,T;\mathbb{L}^{3/2}(\Omega)\right). \tag{4.8}$$

To check this, we shall estimate each term in $\mathcal{F}'$ separately. For example

$$\|\nabla\psi(t)\cdot\tilde{\mathbf{A}}(t)\|_{3/2} \leq \|\nabla\psi(t)\|_2\|\tilde{\mathbf{A}}(t)\|_6 \leq C\|\psi(t)\|_{\mathcal{H}^1}\|\tilde{\mathbf{A}}(t)\|_{\mathbf{H}^1},$$

where $C$ is the Sobolev constant relative to the continuous imbedding of $H^1(\Omega)$ in $L^6(\Omega)$. The other remaining terms can be estimated in the similar way, which confirm (4.8). In the sequel, we define

$$\begin{aligned} \mathcal{F}_\lambda(t) &= \int_0^{t-\lambda} e^{-\mathcal{A}(t-s)}\mathcal{F}'(s,\tilde{u}(s))\, ds \quad \text{for } \lambda \leq t \leq T, \\ \mathcal{F}_\lambda(t) &= 0 \quad \text{for } 0 \leq t \leq \lambda. \end{aligned}$$

For $\lambda > 0$ small, $\mathcal{F}_\lambda$ is well defined and continuous. Indeed we write for $\lambda < t < T$

$$\mathcal{F}_\lambda(t+h) - \mathcal{F}_\lambda(t) = I_1 + I_2$$

where

$$\begin{aligned} I_1 &= \int_0^{t-\lambda} \mathrm{e}^{-\mathcal{A}(t-s)}(\mathrm{e}^{-\mathcal{A}h} - I)\mathcal{F}'(s,\tilde{u}(s))\ \mathrm{d}s \\ I_2 &= \int_{t-\lambda}^{t+h-\lambda} \mathrm{e}^{-\mathcal{A}(t+h-s)}\mathcal{F}'(s,\tilde{u}(s))\ \mathrm{d}s. \end{aligned}$$

By using (4.6), we have

$$\|I_1\|_{\mathbb{H}^1} \le C\int_0^{t-\lambda} (t-s)^{-\gamma}\mathrm{e}^{-\delta(t-s)}\|(\mathrm{e}^{-\mathcal{A}h} - I)\mathcal{F}'(s,\tilde{u}(s))\|_{3/2}\ \mathrm{d}s.$$

Furthermore thanks to (4.8), we can apply Lebesgue theorem to obtain

$$\|I_1\|_{\mathbb{H}^1} \longrightarrow 0 \quad \text{as} \quad h \to 0$$

On the other hand

$$\begin{aligned} \|I_2\|_{\mathbb{H}^1} &\le C\int_{t-\lambda}^{t+h-\lambda} (t+h-s)^{-\gamma}\mathrm{e}^{-\delta(t+h-s)}\|\mathcal{F}'(s,\tilde{u}(s))\|_{3/2}\ \mathrm{d}s \\ &\le C\sup_{0\le t\le T}\|\mathcal{F}'(s,\tilde{u}(s))\|_{3/2}\int_\lambda^{h+\lambda} s^{-\gamma}\mathrm{e}^{-\delta s}\ \mathrm{d}s \end{aligned}$$

and we obtain

$$\|I_2\|_{\mathbb{H}^1} \longrightarrow 0 \quad \text{as} \quad h \to 0$$

When $h \to 0^-$, we obtain a similar estimate and the remaining case $0 \le t \le \lambda$ is trivial. Therefore $\mathcal{F}_\lambda \in C\left(0,T;\mathbb{H}^1(\Omega)\right)$.

Now for $t \in [t_0,t_1] \subset (0,T)$ , we estimate

$$\begin{aligned} \left\|\mathcal{F}_\lambda(t) - \int_0^t \mathrm{e}^{-\mathcal{A}(t-s)}\mathcal{F}'(s,\tilde{u}(s))\ \mathrm{d}s\right\|_{\mathbb{H}^1} &\le \int_{t-\lambda}^t \|\mathrm{e}^{-\mathcal{A}(t-s)}\mathcal{F}'(s,\tilde{u}(s))\|_{\mathbb{H}^1}\ \mathrm{d}s \\ &\le C\int_0^\lambda s^{-\gamma}\mathrm{e}^{-\delta s}\ \mathrm{d}s. \end{aligned}$$

Passing to limit $\lambda \to 0^+$, uniformly for $t_0 \le t \le t_1$ ($t_0$ and $t_1$ are arbitrary), we obtain that the map $\left(t \in (0,T) \longrightarrow \int_0^t \mathrm{e}^{-\mathcal{A}(t-s)}\mathcal{F}'(s,\tilde{u}(s))\ \mathrm{d}s\right)$ is continuous. It remains to show continuity for t=0 and t=T and this is achieved analogously. Therefore

$$\tilde{u} = (\psi,\tilde{\mathbf{A}}) \in C\left(0,T;\mathbb{H}^1(\Omega)\right).$$

**Remark 1** It is not hard to see that the order parameter $\psi$ satisfies moreover the "*maximum principle*": if $\psi_0 \in \mathcal{L}^\infty(\Omega)$ then

$$|\psi(x,t)| \leq \max\left(1, \|\psi_0\|_\infty\right) \quad \text{for all} \quad (x,t) \in \bar{\Omega} \times [0,T]. \tag{4.9}$$

As a consequence of theorem 1, we obtain that the pair $(\psi, \tilde{\mathbf{A}})$ is a weak solution of equations (3.3) and (3.4), while the boundary condition (3.5) is satisfied in some sense of traces.
Observe that theorem 1 includes a comparable result for the pair $u = (\psi, \mathbf{A})$, providing that continuity of $\mathbf{A_H}$ in time occurs. Such a regularity is completely controlled by the continuity of $\mathbf{H}$ in time and the hypothesis $(\mathbf{H_0})$ seems to be only a natural minimal condition for the existence and uniqueness result in theorem 1. However condition $(\mathbf{H_0})$ may be strengthened by requiring that $\mathbf{H} \in C\left(0,T;\mathbf{L}^2(\Omega)\right)$, in this case we obtain the solution $u = (\psi, \mathbf{A}) \in C\left(0,T;\mathbb{H}^1(\Omega)\right)$ and satisfies the gauged TDGL equations (2.3)-(2.4) in a weak sense.

We now concentrate on the regularity of the dependence of the solution $\tilde{u}$ on the initial data $\tilde{u}_0$. As in [9], we can verify as well that the map $\tilde{u}_0 \in X^{1/2} \longrightarrow \tilde{u} \in C\left(0,T;\mathbb{H}^1(\Omega)\right)$ is uniformly Lipschitz continuous on bounded subsets of $X^{1/2}$. This implies the following

**Theorem 2** *The solutions of the abstract initial-value problem (3.12) generate a dynamical process $U = \{U(t,s) : 0 \leq s \leq t \leq T\}$ on $X^{1/2}$ by the definition*

$$\tilde{u}(t) = U(t,s)\tilde{u}(s) \quad \text{for} \quad 0 \leq s \leq t \leq T. \tag{4.10}$$

*Also, for $0 \leq s < t \leq T$, each map $U(t,s) : X^{1/2} \to X^{1/2}$ is compact.*

We omit the proof since the arguments are similar.

**Remark 2** Let us mention that in the particular case, where the magnetic field $\mathbf{H}$ is time constant, the result obtained in [10] concerning asymptotic behavior of the mild solution as $t \to \infty$ remains true, namely the process $U$ becomes a dynamical system $S = \{S(t) : t \geq 0\}$ on $X^{1/2}$, by the definition

$$S(t-s) = U(t,s) \quad \text{for} \quad t \geq s \geq 0.$$

Moreover the dynamical system enjoys the following properties

(i) The functional $E_\omega$ defined in (4.7) is a Liapunov functional for $S$.

(ii) Each $\tilde{u}_0 \in X^{1/2}$ has a relatively compact orbit in $\mathbb{H}^1(\Omega)$.

(iii) The $\omega$-limit set of each $\tilde{u}_0 \in X^{1/2}$ is a nonempty compact connected set of divergence-free equilibria.

(iv) There is a global attractor for $S$.

Here the sense of definitions is borrowed from [17].

## 5 Global Boundedness

In the sequel, we would like to show that in a special case of a smooth magnetic field $\mathbf{H}$, the solutions $\psi$ and $\mathbf{A}$ of the gauged TDGL equations (2.3)-(2.5) become bounded uniformly with respect to $t \geq 0$. In what follows $C$ will denote various constants depending only on the data $\kappa$, $\eta$, $\mathbf{H}$ and the constants entering the equations (2.3)-(2.4), but not on $t$. Also we use the symbol $\partial_t$ to denote the time derivative $\frac{d}{dt}$. Throughout this section, we shall assume that $\mathbf{H}(t) \in \mathbf{H}^1(\Omega)$ for $t \geq 0$ with $\mathbf{H} \in C\left(0, T; \mathbf{L}^2(\Omega)\right)$ for all $T > 0$, $u_0 = (\psi_0, \mathbf{A}_0) \in X^{1/2}$, with $\psi_0 \in \mathcal{L}^\infty(\Omega)$ and $\|\psi_0\|_\infty \leq 1$. Let $u = (\psi, \mathbf{A})$ the corresponding solution of the TDGL equations starting from $u_0$. Remark that since $\mathbf{H}$ is time continuous, it is also the case for the solution $u$. We have the following estimate on the $L^2$-norm of $\psi$ and $\mathbf{A}$.

**Lemma 1** *Assume $\mathbf{H} \in L^\infty\left(0, \infty; \mathbf{L}^2(\Omega)\right)$, then there exists $C > 0$ such that*

$$\|\psi(t)\|_2^2 + \|\mathbf{A}(t)\|_2^2 \leq C\left[\mathrm{e}^{-\lambda_0\omega_0 t}\left(\|\psi_0\|_2^2 + \|\mathbf{A}_0\|_2^2\right) + 1\right] \quad \textit{for all} \quad t \geq 0, \tag{5.1}$$

*where $\omega_0 = \min(1, \omega)$.*

**Proof:** Multiplying the equation (2.3) by the complex conjugate $\psi^*$, integrating over $\Omega$ and taking the real part, we obtain

$$\frac{\eta}{2}\partial_t\|\psi\|_2^2 = -\frac{1}{\kappa^2}\int_{\partial\Omega}\gamma|\psi|^2\,\mathrm{d}\sigma(x) - \left\|\left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)\psi\right\|_2^2 + \|\psi\|_2^2 - \|\psi\|_4^4. \tag{5.2}$$

On the other hand taking the inner product of (2.4) with $\mathbf{A}$, it yields from (2.8) and (2.9)

$$\frac{1}{2}\partial_t\|\mathbf{A}\|_2^2 = -\|\nabla\times\mathbf{A}\|_2^2 - \omega\|\nabla\cdot\mathbf{A}\|_2^2 + \int_\Omega \mathbf{A}\cdot\mathbf{J}_s\,\mathrm{d}x + \int_\Omega \mathbf{H}\cdot(\nabla\times\mathbf{A})\,\mathrm{d}x. \tag{5.3}$$

The last two terms in the right-hand side of (5.3) can be majorized as follows: let $\varepsilon > 0$, replace $\mathbf{J}_s$ in (1.3), so we can apply (4.9) and standard Hölder's and Young's inequalities to obtain

$$\left|\int_\Omega \mathbf{A}\cdot\mathbf{J}_s\,\mathrm{d}x\right| \leq \frac{\varepsilon}{2}\|\mathbf{A}\|_2^2 + \frac{1}{2\varepsilon}\left\|\left(\frac{i}{\kappa}\nabla + \mathbf{A}\right)\psi\right\|_2^2,$$

$$\left|\int_\Omega \mathbf{H}\cdot(\nabla\times\mathbf{A})\ \mathrm{d}x\right| \le \frac{\varepsilon}{2}\|\nabla\times\mathbf{A}\|_2^2+\frac{1}{2\varepsilon}\|\mathbf{H}\|_2^2.$$

Thanks to (2.7), we get

$$\begin{aligned}\frac{1}{2}\partial_t\left(\eta\|\psi\|_2^2+\varepsilon\|\mathbf{A}\|_2^2\right) \ \le\ & -\frac{1}{2}\left\|\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right\|_2^2-\varepsilon\lambda_0\omega_0\|\mathbf{A}\|_{\mathbf{H}^1}^2+\varepsilon^2\|\mathbf{A}\|_{\mathbf{H}^1}^2\\ & +\|\psi\|_2^2+\frac{1}{2}\|\mathbf{H}\|_2^2. \end{aligned} \tag{5.4}$$

Set $\zeta(t)=\eta\|\psi(t)\|_2^2+\varepsilon\|\mathbf{A}(t)\|_2^2$. Since $\mathbf{H}\in L^\infty\left(0,\infty;\mathbf{L}^2(\Omega)\right)$, it follows by choosing $0<\varepsilon<\frac{\lambda_0\omega_0}{2}$ that

$$\partial_t\zeta(t)+\lambda_0\omega_0\zeta(t)\le C \quad \text{for all} \quad t\ge 0.$$

Thus after substituting in inequality (2.10), we obtain

$$\zeta(t)\le \mathrm{e}^{-\lambda_0\omega_0 t}\zeta(0)+\frac{C}{\lambda_0\omega_0} \quad \text{for all} \quad t\ge 0.$$

This concludes the proof of the lemma.

The next theorem establishes the $H^1$-norm global boundedness of the solutions $\psi$ and $\mathbf{A}$ of the TDGL equations (2.3)-(2.6).

**Theorem 3** *Provided $\mathbf{H}\in W^{1,\infty}\left(0,\infty;\mathbf{L}^2(\Omega)\right)$, there exists $C>0$ such that*

$$\|\psi(t)\|_{\mathcal{H}^1}^2+\|\mathbf{A}(t)\|_{\mathbf{H}^1}^2\le C\left[\mathrm{e}^{-\varepsilon t}\left(\|\psi_0\|_{\mathcal{H}^1}^2+\|\mathbf{A}_0\|_{\mathbf{H}^1}^2\right)+1\right] \text{ for all } t\ge 0, \tag{5.5}$$

*where $\varepsilon>0$ is small enough.*

**Proof:** First we estimate the $H^1$-norm of $\mathbf{A}$. Taking the inner product of (2.4) with $\partial_t\mathbf{A}$, we have

$$\begin{aligned}\frac{1}{2}\partial_t\left(\|\nabla\times\mathbf{A}\|_2^2+\omega\|\nabla\cdot\mathbf{A}\|_2^2\right) = & -\int_\Omega\partial_t\mathbf{H}\cdot(\nabla\times\mathbf{A})\ \mathrm{d}x+\partial_t\left(\int_\Omega\mathbf{H}\cdot(\nabla\times\mathbf{A})\ \mathrm{d}x\right)\\ & -\|\partial_t\mathbf{A}\|_2^2+\int_\Omega\mathbf{J}_s\cdot\partial_t\mathbf{A}\ \mathrm{d}x. \end{aligned} \tag{5.6}$$

Using similar arguments as above, we get

$$\begin{aligned}\left|\int_\Omega\partial_t\mathbf{H}\cdot(\nabla\times\mathbf{A})\ \mathrm{d}x\right| &\le \frac{\varepsilon}{2}\|\nabla\times\mathbf{A}\|_2^2+\frac{1}{2\varepsilon}\|\partial_t\mathbf{H}\|_2^2,\\ \left|\int_\Omega\mathbf{J}_s\cdot\partial_t\mathbf{A}\ \mathrm{d}x\right| &\le \frac{1}{2}\|\partial_t\mathbf{A}\|_2^2+\frac{1}{2}\left\|\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right\|_2^2. \end{aligned}$$

Thus

$$\frac{1}{2}\partial_t\left(\|\nabla\times\mathbf{A}\|_2^2+\omega\|\nabla\cdot\mathbf{A}\|_2^2-2\int_\Omega \mathbf{H}\cdot(\nabla\times\mathbf{A})\,\mathrm{d}x\right)$$
$$\le\frac{1}{2}\left\|\left(\frac{i}{\kappa}\nabla+\mathbf{A}\right)\psi\right\|_2^2+\frac{\varepsilon}{2}\|\nabla\times\mathbf{A}\|_2^2+\frac{1}{2\varepsilon}\|\partial_t\mathbf{H}\|_2^2. \tag{5.7}$$

Multiplying (5.7) by $\varepsilon$, $0<\varepsilon<1$ and adding estimate (5.4) yield

$$\frac{1}{2}\partial_t\left[\eta\|\psi\|_2^2+\varepsilon(\|\mathbf{A}\|_2^2+\|\nabla\times\mathbf{A}\|_2^2+\omega\|\nabla\cdot\mathbf{A}\|_2^2)-2\varepsilon\int_\Omega \mathbf{H}\cdot(\nabla\times\mathbf{A})\,\mathrm{d}x\right]$$
$$\le-\varepsilon\lambda_0\omega_0\|\mathbf{A}\|_{\mathbf{H}^1}^2+\frac{3}{2}\varepsilon^2\|\mathbf{A}\|_{\mathbf{H}^1}^2+\|\psi\|_2^2+\frac{1}{2}\left(\|\mathbf{H}\|_2^2+\|\partial_t\mathbf{H}\|_2^2\right), \tag{5.8}$$

so by putting

$$\begin{aligned}\vartheta(t) &= \eta\|\psi(t)\|_2^2+\varepsilon\left(\|\mathbf{A}(t)\|_2^2+\|\nabla\times\mathbf{A}(t)\|_2^2+\omega\|\nabla\cdot\mathbf{A}(t)\|_2^2\right)\\ &\quad -2\varepsilon\int_\Omega \mathbf{H}(t)\cdot(\nabla\times\mathbf{A}(t))\,\mathrm{d}x,\end{aligned}$$

we deduce

$$\begin{aligned}\partial_t\vartheta(t)+\varepsilon\vartheta(t) &\le -2\varepsilon\lambda_0\omega_0\|\mathbf{A}\|_{\mathbf{H}^1}^2+\varepsilon^2(4+\omega_1)\|\mathbf{A}\|_{\mathbf{H}^1}^2+(2+\varepsilon\eta)\|\psi\|_2^2\\ &\quad +2\|\mathbf{H}\|_2^2+\|\partial_t\mathbf{H}\|_2^2,\end{aligned} \tag{5.9}$$

with $\omega_1=\max(1,\omega)$, which with the assumption $\mathbf{H}\in W^{1,\infty}(0,\infty;\mathbf{L}^2(\Omega))$ implies

$$\partial_t\vartheta(t)+\varepsilon\vartheta(t)\le C\quad\text{for all}\quad t\ge 0,$$

provided $0<\varepsilon<\frac{2\lambda_0\omega_0}{4+\omega_1}$. Hence Gronwall's inequality (2.10) shows

$$\vartheta(t)\le \mathrm{e}^{-\varepsilon t}\vartheta(0)+\frac{C}{\epsilon}\quad\text{for all}\quad t\ge 0.$$

Therefore

$$\|\psi(t)\|_2^2+\|\mathbf{A}(t)\|_{\mathbf{H}^1}^2\le C\left[\mathrm{e}^{-\varepsilon t}(\|\psi_0\|_2^2+\|\mathbf{A}_0\|_{\mathbf{H}^1}^2)+1\right]\ \text{for all } t\ge 0. \tag{5.10}$$

On the other hand, to estimate the $H^1$-norm of $\psi$, we make use of the energy type functional $E_\omega$ introduced in (4.7). Since $\psi$ and $\mathbf{A}$ satisfy equations (2.3)-(2.4), the time derivative of $E_\omega$ is

$$\begin{aligned}\partial_t E_\omega(t) &= -2\int_\Omega\left[\eta|\partial_t\psi-i\kappa\omega\psi(\nabla\cdot\mathbf{A})|^2+|\partial_t\mathbf{A}|^2+\omega^2|\nabla(\nabla\cdot\mathbf{A})|^2\right]\,\mathrm{d}x\\ &\quad -2\int_\Omega \partial_t\mathbf{H}\cdot(\nabla\times\mathbf{A}-\mathbf{H})\,\mathrm{d}x.\end{aligned}$$

This implies

$$\partial_t E_\omega(t) \leq -2 \int_\Omega \partial_t \mathbf{H} \cdot (\nabla \times \mathbf{A} - \mathbf{H}) \, \mathrm{d}x. \tag{5.11}$$

Now adding estimates (4.7) and (5.11), thanks to Hölder's and Young's inequalities, so it follows

$$\partial_t E_\omega(t) + E_\omega(t) \leq \left\| \left( \frac{i}{\kappa} \nabla + \mathbf{A} \right) \psi \right\|_2^2 + C \left( \|\mathbf{A}\|_{\mathbf{H}^1}^2 + \|\mathbf{H}\|_2^2 + \|\partial_t \mathbf{H}\|_2^2 + 1 \right), \tag{5.12}$$

therefore by putting $\xi(t) = E_\omega(t) + \eta \|\psi(t)\|_2^2$, we derive from (5.2) and (5.12)

$$\partial_t \xi(t) + \xi(t) \leq C(\|\mathbf{A}\|_{\mathbf{H}^1}^2 + 1) \quad \text{for all} \quad t \geq 0.$$

Once more, Gronwall's inequality (2.10) yields

$$\xi(t) \leq \mathrm{e}^{-t} \left[ \xi(0) + C \int_0^t \mathrm{e}^s (\|\mathbf{A}(s)\|_{\mathbf{H}^1}^2 + 1) \, \mathrm{d}s \right] \quad \text{for all} \quad t \geq 0,$$

and by (5.10), we infer that

$$\xi(t) \leq C \left[ \mathrm{e}^{-\varepsilon t} \left( \|\psi_0\|_{\mathcal{H}^1}^2 + \|\mathbf{A}_0\|_{\mathbf{H}^1}^2 \right) + 1 \right] \quad \text{for all} \quad t \geq 0.$$

Consequently by replacing $E_\omega$ in (4.7) and taking in mind (5.10), we conclude

$$\|\nabla \psi(t)\|_2^2 \leq C \left[ \mathrm{e}^{-\varepsilon t} (\|\psi_0\|_{\mathcal{H}^1}^2 + \|\mathbf{A}_0\|_{\mathbf{H}^1}^2) + 1 \right],$$

which proves theorem 2.

**Remark 3** Theorem 3 remains true also for the pair $\tilde{u} = (\psi, \tilde{\mathbf{A}})$ of solutions of the reduced homogeneous problem (3.3)- (3.4). On the other hand, we can use equation (3.15) to improve the regularity of the dependence of $\tilde{u}$ on the initial data $\tilde{u}_0$; that is the set $\{U(t,0)\tilde{u}_0 : t \geq 0, \ \|\tilde{u}_0\|_{\mathbb{H}^1} \leq R\}$ $(R > 0)$, is relatively compact in $\mathbb{H}^1(\Omega)$.

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**Author:**

Fouzi Zaouch
Universität Rostock
Fachbereich Mathematik
18051 Rostock
Germany

e-mail: fouzi.zaouch@mathematik.uni-rostock.de