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# B-Nearness on Boolean frames

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ABSTRACT. In this paper we define and study BN-proximity, B-nearness, B-farness, B-smallness and B-covering on Boolean frames and investigate relation between them. Also we define and investigate some properties of B-nearness on frames and subframes. Then we define complete nearness space by given B-near frame.

KEY WORDS. BN-Proximity, B-Nearness, B-Covering, B-Smallness, B-Farness, Contigual, Uniform, graded, Complete.

## 1 Introduction

Nearness on space introduced by Herrlich on 1974. Topics developed in this paper are based on the work of Banaschewski and Dube. We define and study BN-proximity, B-nearness, B-farness, B-smallness and B-covering on Boolean frames and we investigate relation between them.

Also we investigate some properties of B-nearness on frames and subframes. Then we have shown that by given B-Nearness frame we can define a complete nearness space.

## 2 Background

**Definition 1** Let $X$ be a set and let $\xi$ be a subset of $P^2X$. Consider the following axioms:

$(N1)$ If $\mathcal{A} << \mathcal{B}$ and $\mathcal{B} \in \xi$ then $\mathcal{A} \in \xi$. Where $\mathcal{A} << \mathcal{B}$ iff $\forall A \in \mathcal{A} \exists B \in \mathcal{B}, A \supseteq B$;

$(N2)$ If $\bigcap \mathcal{A} \neq \emptyset$ then $\mathcal{A} \in \xi$;

$(N3)$ $\emptyset \neq \xi \neq P^2X$;

$(N4)$ If $(\mathcal{A} \bigvee \mathcal{B}) \in \xi$ then $\mathcal{A} \in \xi$ or $\mathcal{B} \in \xi$, where $\mathcal{A} \bigvee \mathcal{B} := \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\}$;

(N5) If $\{cl_\xi A | A \in \mathcal{A}\} \in \xi$ then $\mathcal{A} \in \xi$, where $cl_\xi A := \{x \in X | \{A, \{x\}\} \in \xi\}$.

$\xi$ is called nearness structure on X iff $\xi$ satisfying to above conditions, and the pair $(X, \xi)$ is called a nearness space -shortly a N-space- iff $\xi$ is a nearness structure on $X$.

If $(X, \xi)$ and $(Y, \eta)$ are N-spaces then a function $f : X \longrightarrow Y$ is called a nearness preserving map from $(X, \xi)$ to $(Y, \eta)$ iff $\mathcal{A} \in \xi$ implies $f\mathcal{A} \in \eta$, where $f\mathcal{A} := \{f[A] : A \in \mathcal{A}\}$.

Let $(X, \xi)$ be a N-space, then a subset $\mathcal{A}$ of PX is called a $\xi$-*cluster* iff $\mathcal{A}$ is maximal element of the set $\xi \setminus \{\emptyset\}$, ordered by inclusion. We call $(X, \xi)$ complete iff every $\xi$-*cluster* contains an element $\{x\}$ for some $x \in X$.

$(X^*, \xi^*)$ is a completion of $(X, \xi)$ where $\xi^* = \{\Omega \subset PX^* | \cup \{\cap \omega | \omega \in \Omega\} \in \xi\}$ and $X^*$ denotes the set of all $\xi$-*clusters*.

**Definition 2** A *frame* is a complete lattice satisfying the special distribution law,

$(IFD1)$ $\forall\, a \in L, \forall\, S \subseteq L,\, a \wedge \bigvee S = \bigvee \{a \wedge x | x \in S\}$;

and it is called a Boolean frame if it is complementary.

Note, that in this case each element has unique complement, i.e.

$\forall\, a \in L\ \exists!\ a' \in L$ s.t. $a \wedge a' = 0$ and $a \vee a' = 1$

Therefore Boolean frame satisfying in

$(IFD2)$ $\forall\, a \in L, \forall\, S \subseteq L,\, a \vee \bigwedge S = \bigwedge \{a \vee x | x \in S\}$.

Frame homomorphisms between Boolean frames preserve top, bottom (denoted by 1 and 0 respectively) meets, joins and complementary.

## 3 B-Nearness and BN-proximity on frames

**Definition 3** Let $L$ be a Boolean frame and $A$, $B$ are subsets of $L$.

$$secA = \{x \in L | \forall a \in A, x \wedge a \neq 0\}\,;$$
$$stackA = \{x \in L | \exists a \in A \text{ s.t. } a \leq x\}\,;$$
$$A \bigvee B = \{a \vee b | a \in A, b \in B\}\,;$$
$$A \bigwedge B = \{a \wedge b | a \in A, b \in B\}\,;$$
$$A' = \{a' | a \in A\}\,;$$
$$st(x, A) = \bigvee \{a \in A | a \wedge x \neq 0\}\,;$$
$$st(x, A)^d = \bigwedge \{a \in A | a \vee x \neq 1\}\,;$$

The partial order defined by setting $a \leq b$ iff $b = a \vee b$;

$$A << B \text{ iff } \forall a \in A\ \exists b \in B \text{ s.t. } b \leq a;\ (A \text{ corefines } B)$$
$$A \prec B \text{ iff } \forall a \in A\ \exists b \in B \text{ s.t. } a \leq b.\ (A \text{ refines } B)$$

**Proposition 1** Let $L$ be a Boolean frame and $A$, $B$ are subsets of $L$. Then we have following statements:

(1) $A << B$ iff $A' \prec B'$;

(2) $stack(A \cup B) = stackA \cup stackB$;

(3) $stack\emptyset = \emptyset$;

(4) $stackA = sec^2A$;

(5) $sec^3A = secA$;

**Proof:**

(1) $A << B$ iff $\forall a \in A$, $\exists b \in B$ s.t. $b \leq a$ so $a' \leq b'$ i.e. $\forall a' \in A'$, $\exists b' \in B'$ s.t. $a' \leq b'$ iff $A' \prec B'$.

(2) and (3) obviously hold.

(4) Let $b \in stackA$ i.e. $\exists a \in A$ s.t. $a \leq b$. Let $c$ be arbitrary member of $secA$ so $\forall a \in A$, $c \wedge a \neq 0$ and since $a \leq b$, it implies $c \wedge b \neq 0$ therefore we have $\forall c \in secA$, $b \wedge c \neq 0$ i.e. $b \in sec^2A$ so $stackA \subseteq sec^2A$.
Now let $b \in sec^2A$, if $b \notin stackA$, then $\forall a \in A$, $a \nleq b$ so $\forall a \in A$, $b' \wedge a \neq 0$ i.e. $b' \in secA$ so $b \notin sec^2A$ which is contradiction so $b \in stackA$ i.e. $sec^2A \subseteq stackA$.
Therefore $stackA = sec^2A$.

(5) Let $c \in sec^3A = stack(secA)$ so $\exists d \in secA$ s.t. $d \leq c$ but $\forall a \in A$, $d \wedge a \neq 0$ therefore $\forall a \in A$, $c \wedge a \neq 0$ i.e. $c \in secA$ so $sec^3A \subseteq secA$. And obviously, $secA \subseteq stack(secA)$ so $sec^3A = secA$.

**Definition 4** Let $L$ be a Boolean frame. The relation $\delta$ satisfying in the following conditions:

(BP0) $x\delta y$ implies $y\delta x$;

(BP1) $x \leq y$ and $x\delta z$ imply $y\delta z$;

(BP2) $x \wedge y \neq 0$ implies $x\delta y$;

(BP3) $x\delta y$ implies $x \neq 0$;

(BP4) $x\delta(y \vee z)$ implies $x\delta y$ or $x\delta z$;

(BP5) For every $x \in L$ we have $x = \bigwedge \{y \in L | x \leq y, x\overline{\delta} y'\}$.
Relation $\delta$ is called *BN-proximity* on $L$ and $(L, \delta)$ is *BN-proximal frame*.
If additionally $\delta$ satisfies in (BP6),

(BP6) $x\overline{\delta}y$ implies there exist $z \in L$ such that $x\overline{\delta}z$ and $z'\overline{\delta}y$; ( $x\overline{\delta}y$ means $x$ and $y$ are not in relation.)
Then $\delta$ is called *B-proximity* on $L$ and $(L, \delta)$ is *B-proximal frame*.

Let $(L, \delta_1)$ and $(M, \delta_2)$ be two B-proximal frames.

A frame homomorphism $f : L \to M$ is called frame B-proximal homomorphism iff

$$x\overline{\delta_1}y \text{ implies } f(x)\overline{\delta_2}f(y)$$

We denote the corresponding category by **BProxFrm**.

**Definition 5** Let $L$ be a Boolean frame and $\xi$ be a subset of $PL$:

(BN1) If $A << B$ and $B \in \xi$ then $A \in \xi$;

(BN2) If $\bigwedge A \neq 0$ then $A \in \xi$;

(BN3) $\emptyset \neq \xi \neq PL$;

(BN4) If $(A \bigvee B) \in \xi$ then $A \in \xi$ or $B \in \xi$;

Then $\xi$ is called *preB-nearness* on $L$, and the pair $(L, \xi)$ is called *preB-nearness frame* iff it satisfies in (BN1)-(BN3).
$\xi$ is called *semiB-nearness* on $L$, and the pair $(L, \xi)$ is called *semiB-nearness frame* iff it satisfies in (BN1)-(BN4).
$\xi$ is called *B-nearness* on $L$, and the pair $(L, \xi)$ is called *B-nearness frame* iff it additionally satisfies:

(BN5) For each $x \in L$, $x = \bigwedge \{y \in L | x \leq st(y, A)^d, \text{ for some } A \notin \xi\}$.

Let $(L, \xi)$ and $(M, \eta)$ be two B-nearness frames.
A frame homomorphism $f : L \to M$ is called B-nearness homomorphism iff
$A \in \eta \Rightarrow f^{-1}(A) \in \xi$.

The corresponding category denoted by **BNFrm**.

**Proposition 2** Let $L$ be a Boolean frame, $y \in L$ and $A \subseteq L$ where $\bigwedge A = 0$ then $st(y, A)^d \leq y$.

**Proof:** We know for every $y \in L$

$$y = y \vee 0 = y \vee (\bigwedge \{z | z \in A\}) = \bigwedge \{y \vee z | z \in A\}.$$

And since $L$ is Boolean if for all $z \in A$, $y \vee z = 1$ then $\forall z \in A$, $y' \leq z$ so $y' \leq \bigwedge A$ i.e. $y' = 0$ so $y = 1$ and then $y = \wedge\emptyset$ so we have

$$\begin{aligned} y &= \bigwedge \{y \vee z | y \vee z \neq 1, z \in A\} \\ &= y \vee (\bigwedge \{z \in A | z \vee y \neq 1\}) \\ &= y \vee st(y, A)^d \;\Rightarrow\; st(y, A)^d \leq y \end{aligned}$$

**Proposition 3** Let $L$ be a Boolean frame and $\xi$ is a preB-nearness on $L$ then $x \leq st(y, A)^d$ for some $A \notin \xi$ iff $x \leq y$ and $\{y', x\} \notin \xi$.

**Proof:** Let $x \leq st(y, A)^d$ for some $A \notin \xi$. Since $A \notin \xi$ by (BN2), $\bigwedge A = 0$ and by Proposition 2, $st(y, A)^d \leq y$ so $x \leq \bigwedge \{z | z \vee y \neq 1, z \in A\} \leq y$. Now for every $z \in A$ either $z \vee y \neq 1$ or $z \vee y = 1$. If $z \vee y \neq 1$ then $x \leq z$ and if $z \vee y = 1$ then $y' \leq z$ i.e. $A << \{y', x\}$ and since $A \notin \xi$ by (BN1), $\{y', x\} \notin \xi$.

Conversely, let $x \leq y$ and $\{y', x\} \notin \xi$. For $A = \{y', x\}$, if $y = 1$ then $st(y, A)^d = 1$ and if $y \neq 1$ then $st(y, A)^d = x$. In any case $x \leq st(y, A)^d$ for some $A \notin \xi$.

**Remark 1** Let $(L, \xi)$ be a semiB-nearness frame then $\xi$ is B-nearness frame iff it satisfies in the following condition:

(BN5$'$) For each $x \in L$, $x = \bigwedge \{y \in L | x \leq y,\ and\ \{y', x\} \notin \xi\}$.

**Theorem 1** Let $L$ be a Boolean frame. Set of all preB-nearness on $L$ ordered by set inclusion is a lattice when its bottom is $\{A \subset L | \bigwedge A \neq 0\}$ and its top is $\{A \subset L | 0 \notin A\}$ .

**Remark 2** If $(X, \xi)$ is a B-nearness frame then the relation $\delta$ on $L$ defined by

$$x \delta y \text{ iff } \{x, y\} \in \xi$$

is a BN-proximity.

**Proof:** We show that $\delta$ satisfying in the (BP0)-(BP5).

To (BP0): Let $x \delta y$ then $\{x, y\} \in \xi$ and equivalently $\{y, x\} \in \xi$ i.e. $y \delta x$.

To (BP1): Let $x \leq y$ and $x \delta z$ i.e. $\{x, z\} \in \xi$ since $\{y, z\} << \{x, z\}$ by (BN1) we have $\{y, z\} \in \xi$ i.e. $y \delta z$.

To (BP2): Let $x \wedge y \neq 0$ so by (BN2), $\{x, y\} \in \xi$ i.e. $x \delta y$.

To (BP3): Let $x \delta y$ i.e. $\{x, y\} \in \xi$. If $x = 0$ then for every $A \subset L$, $A << \{x, y\}$ and by (BN1), $A \in \xi$ i.e. $\xi = L$ which is contradiction to (BN3). So $x \neq 0$.

To (BP4): Let $x \delta (y \vee z)$ i.e. $\{x, (y \vee z)\} \in \xi$ but we have
$\{x, y\} \vee \{x, z\} = \{x, x \vee z, y \vee x, y \vee z\} << \{x, (y \vee z)\}$ so by (BN1), $\{x, y\} \vee \{x, z\} \in \xi$ and by (BN4), $\{x, y\} \in \xi$ or $\{x, z\} \in \xi$ i.e. $x \delta y$ or $x \delta z$.

To (BP5): By (BN5′) we have for every $x \in L$, $x = \bigwedge\{y \in L|x \leq y,\ and\ \{y', x\} \notin \xi\}$ i.e. $x = \bigwedge\big\{y \in L|x \leq y,\ \text{and}\ x\bar{\delta}y'\big\}$.

Therefore $\delta$ is an *BN-proximity* on $L$.

## 4 B-Farness, B-Smallness and B-Covering on frames

**Definition 6** Let $L$ be a Boolean frame let $\overline{\xi}$ be a subset of $PL$ satisfying the following conditions:

(BF1) If $A << B$ and $A \in \overline{\xi}$ then $B \in \overline{\xi}$;

(BF2) If $A \in \overline{\xi}$ then $\bigwedge A = 0$ ;

(BF3) $\emptyset \neq \overline{\xi} \neq PL$;

(BF4) If $A \in \overline{\xi}$ and $B \in \overline{\xi}$ then $(A \bigvee B) \in \overline{\xi}$;

Then $\overline{\xi}$ is called *B-farness* on L, and the pair $(L, \overline{\xi})$ is called a *B-farness frame* iff it additionally satisfies:

(BF5) For each $x \in L$, $x = \bigwedge\big\{y \in L|x \leq st(y, A)^d,\ for\ some\ A \in \overline{\xi}\big\}$.

**Definition 7** Let $L$ be a Boolean frame let $\gamma$ be a subset of $PL$ satisfying the following conditions:

(BS1) If $A << B$ and $A \in \gamma$ then $B \in \gamma$;

(BS2) For $0 \in L$, $\{0\} \in \gamma$;

(BS3) $\emptyset \neq \gamma \neq PL$;

(BS4) If $A \cup B \in \gamma$ then $A \in \gamma$ or $A \in \gamma$;

Then $\gamma$ is called *B-smallness* on L, and the pair $(L, \gamma)$ is called a *B-smallness frame* iff it additionally satisfies:

(BS5) For each $x \in L$, $x = \bigwedge\big\{y \in L|x \leq st(y, A)^d,\ \text{for some}\ secA \notin \gamma\big\}$.

**Definition 8** [2] Let $L$ be a Boolean frame let $\mu$ be a subset of $PL$ satisfying the following conditions:

(BC1) If $A \prec B$ and $A \in \mu$ then $B \in \mu$;

(BC2) If $A \in \mu$ then $\bigvee A = 1$ ;

(BC3) $\emptyset \neq \mu \neq PL$;

(BC4) If $A \in \mu$ and $B \in \mu$ then$(A \bigwedge B) \in \mu$;

Then $\mu$ is called *B-covering* on L, and the pair $(L, \mu)$ is called a *B-covering frame* iff it additionally satisfies:

(BC5) For each $x \in L$, $x = \bigvee \{y \in L | st(y, A) \leq x,$ for some $A \in \mu\}$.

Let $(L, \mu_1)$ and $(M, \mu_2)$ be two B-covering frames, a frame homomorphism $f : L \to M$ is called B-covering homomorphism iff $A \in \mu_1 \Rightarrow \{f(a) : a \in A\} =: f(A) \in \mu_2$.

We denote the corresponding category with **BCFrm**.

**Proposition 4** Let $L$ be a Boolean frame and $\xi$ be a B-nearness on $L$. Then

(i) $\overline{\xi} = \{A \subseteq L | A \notin \xi\}$ is the B-farness structure on $L$ induced by $\xi$;

(ii) $\mu = \left\{A \subseteq L | A' \in \overline{\xi}\right\}$ is the B-covering structure on $L$ induced by $\xi$;

(iii) $\gamma = \{A \subset L | \forall B \in \mu, B \cap stackA \neq \emptyset\}$ is the B-smallness structure on $L$ induced by $\xi$.

**Proof:**

(i) We prove that $\overline{\xi}$ is a B-farness on $L$.

To (BF1): Let $A << B$ and $A \in \overline{\xi}$ then $A \notin \xi$ so by (BN1), $B \notin \xi$ i.e. $B \in \overline{\xi}$.

To (BF2): If $A \in \overline{\xi}$ i.e. $A \notin \xi$ so $\wedge A = 0$.

To (BF3): Since $\xi \neq \emptyset$, $\overline{\xi} \neq PL$. And since $\xi \neq PL$, $\overline{\xi} \neq \emptyset$.

To (BF4): Let $A \in \overline{\xi}$ and $B \in \overline{\xi}$ i.e. $A \notin \xi$ and $B \notin \xi$ so by (BN4), $A \vee B \notin \xi$ i.e. $A \vee B \in \overline{\xi}$.

To (BF5): Let $x \in L$, we know $x = \bigwedge \left\{y \in L | x \leq st(y, A)^d,\right.$ for some $\left.A \notin \xi\right\}$ i.e. $x = \bigwedge \left\{y \in L | x \leq st(y, A)^d,\right.$ for some $\left.A \in \overline{\xi}\right\}$.

So $\overline{\xi}$ is a B-farness on $L$ which induced by $\xi$.

(ii) We prove that $\mu$ is a B-covering on $L$.

To (BC1): Let $A \prec B$ and $A \in \mu$ i.e. $A' << B'$ and $A' \in \overline{\xi}$ so by (BF1), $B' \in \overline{\xi}$ i.e. $B \in \mu$.

To (BC2): Let $A \in \mu$ i.e. $A' \in \overline{\xi}$ so by (BF2), $\wedge A' = 0$ so $\vee A = 1$.

To (BC3): Since $\overline{\xi} \neq \emptyset$, $\mu \neq \emptyset$ and since $\overline{\xi} \neq PL$, $\mu \neq PL$.

To (BC4): Let $A \in \mu$ and $B \in \mu$ i.e. $A' \in \overline{\xi}$ and $B' \in \overline{\xi}$ so by (BF4), $A' \vee B' \in \overline{\xi}$ i.e. $(A \wedge B)' \in \overline{\xi}$ so $A \wedge B \in \mu$.

To (BC5): Let $x \in L$, we know $x' = \bigwedge \left\{y' \in L | x' \leq st(y', A')^d,\right.$ for some $\left.A' \in \overline{\xi}\right\}$ so $x = \bigvee \left\{y \in L | x' \leq st(y', A')^d,\right.$ for some $\left.A' \in \overline{\xi}\right\}$.

But $x' \leq st(y', A')^d$ means $x' \leq \bigwedge \{a' \in A' | a' \vee y' \neq 1\}$ i.e.

$\bigvee \{a \in A | a \wedge y \neq 0\} \leq x$ i.e. $st(y, A) \leq x$ so

$x = \bigvee \left\{y \in L | st(y, A) \leq x,\right.$ for some $\left.A' \in \overline{\xi}\right\}$ i.e.

$x = \bigvee\{y \in L | st(y, A) \leq x, \text{ for some } A \in \mu\}$.

So $\mu$ is a B-covering on $L$ which induced by $\xi$.

(iii) We prove that $\gamma$ is a B-smallness on $L$.

To (BS1): If $A << B$ and $A \in \gamma$ then for every $D \in \mu$, $D \cap stackA \neq \emptyset$ i.e. for arbitrary $D \in \mu$, there exists $d \in D$ s.t. $a \leq d$ for some $a \in A$. Also we know $A << B$ so there exists $b \in B$ s.t. $b \leq a$ therefore $b \leq a \leq d$ i.e. $d \in stackB$ so $D \cap stackB \neq \emptyset$ i.e. $B \in \gamma$.

To (BS2): We know $stack\,\{0\} = \{x \in L | 0 \leq x\} = L$ and obviously for every$B \in \mu$, $B \cap L \neq \emptyset$. So $\{0\} \in \gamma$.

To (BS3): Since $\{0\} \in \gamma$ so $\gamma \neq \emptyset$.

We know $stack\emptyset = \emptyset$ and for every $B \in \mu$, $B \cap \emptyset = \emptyset$ so $\emptyset \notin \gamma$ i.e. $\gamma \neq PL$.

To (BS4): Let $A \cup B \in \gamma$ so for every $D \in \mu$, $D \cap stack(A \cup B) \neq \emptyset$ i.e. $D \cap (stackA \cup stackB) \neq \emptyset$ so $(D \cap stackA) \cup (D \cap stackB) \neq \emptyset$ i.e. either $D \cap stackA \neq \emptyset$ or $D \cap stackB \neq \emptyset$.

$$\text{Let for} \quad D_1 \in \mu, \quad D_1 \cap stackA = \emptyset \quad \text{and} \quad D_1 \cap stackB \neq \emptyset \tag{1}$$
$$\text{Let for} \quad D_2 \in \mu, \quad D_2 \cap stackA \neq \emptyset \quad \text{and} \quad D_2 \cap stackB = \emptyset \tag{2}$$

Since $D_1 \in \mu$ and $D_2 \in \mu$, by (BC4), $D_1 \wedge D_2 \in \mu$.

So either $(D_1 \wedge D_2) \cap stackA \neq \emptyset$ or $(D_1 \wedge D_2) \cap stackB \neq \emptyset$.

If $(D_1 \wedge D_2) \cap stackA \neq \emptyset$, there exists $d_1 \wedge d_2$ where $d_1 \in D_1$ and $d_2 \in D_2$ s.t. $a \leq (d_1 \wedge d_2)$ for some $a \in A$ therefore $a \leq d_1$ and $a \leq d_2$, i.e. $D_1 \cap stackA \neq \emptyset$ and $D_2 \cap stackA \neq \emptyset$ which is contradiction to (1). Similarly if $(D_1 \wedge D_2) \cap stackB \neq \emptyset$, we have $D_1 \cap stackB \neq \emptyset$ and $D_2 \cap stackB \neq \emptyset$ which is contradiction to (2).

So either for all $D \in \mu$, $D \cap stackA \neq \emptyset$ or for all $D \in \mu$, $D \cap stackB \neq \emptyset$ i.e. either $A \in \gamma$ or $B \in \gamma$.

To (BS5): Let $x \in L$ so $x = \bigwedge\left\{y \in L | x \leq st(y, A)^d, \text{ for some } A \in \overline{\xi}\right\}$ i.e.

$x = \bigwedge\left\{y \in L | x \leq st(y, A)^d, \text{ for some } A' \in \mu\right\}$.

If $A' \in \mu$ then $secA \notin \gamma$ since $A' \cap stack(secA) = A' \cap secA = \emptyset$.

And if $secA \notin \gamma$ then there is $B \in \mu$ s.t. $B \cap stack(secA) = \emptyset$, i.e. $B \cap secA = \emptyset$, i.e. for every $b \in B$ there is $a \in A$ s.t. $b \wedge a = 0$ so $b \leq a'$ therefore we have $B \prec A'$ and by (BC1) it implies $A' \in \mu$.

So equivalently we have $x = \bigwedge\left\{y \in L | x \leq st(y, A)^d, \text{ for some } secA \notin \gamma\right\}$.

Therefore $\gamma$ is a B-smallness on $L$ which induced by $\xi$.

**Proposition 5** Let $L$ be a Boolean frame and $\xi$, $\overline{\xi}$, $\gamma$ and $\mu$ be respectively B-nearness, B-farness, B-smallness and B-covering structures induced by each other on $L$. Then following relations are hold

(1) $A \in \xi$ iff $A \notin \overline{\xi}$;

(2) $A \in \overline{\xi}$ iff $A \notin \xi$;

(3) $A \in \overline{\xi}$ iff $A' \in \mu$;

(4) $A \in \mu$ iff $A' \in \overline{\xi}$;

(5) $A \in \mu$ iff $\forall\ B \in \xi\ A \cap secB \neq \emptyset$;

(6) $A \in \xi$ iff $\forall\ B \in \mu$, $B \cap secA \neq \emptyset$;

(7) $A \in \gamma$ iff $\forall\ B \in \mu$, $B \cap stackA \neq \emptyset$;

(8) $A \in \mu$ iff $\forall B \in \gamma$, $A \cap stackB \neq \emptyset$;

(9) $A \in \xi$ iff $secA \in \gamma$;

(10) $A \in \gamma$ iff $secA \in \xi$;

(11) $A \in \gamma$ iff $\forall B \in \overline{\xi}\ \exists a \in A\ \exists b \in B$, $a \wedge b = 0$;

(12) $A \in \overline{\xi}$ iff $\forall B \in \gamma$, $\exists a \in A\ \exists b \in B$, $a \wedge b = 0$.

**Proof:**

By Proposition 4, (1) - (4) is clear.

(5) Let $A \in \mu$ then $A' \in \overline{\xi}$ and let $B \in \xi$ we prove that $A \cap secB \neq \emptyset$ i.e. $\exists\ a \in A$ s.t. $a \in secB$ i.e. $\exists\ a \in A$ s.t. $\forall\ b \in B$, $a \wedge b \neq 0$. If not so $\forall a \in A\ \exists\ b \in B$ s.t. $a \wedge b = 0$ i.e. $b \leq a'$ therefore $\forall a' \in A'\ \exists\ b \in B$ s.t. $b \leq a'$ i.e. $A' << B$ and since $A' \in \overline{\xi}$ by (BF1) $B \in \overline{\xi}$ which is a contradiction.

Conversely, let $\forall\ B \in \xi\ A \cap secB \neq \emptyset$. If $A \notin \mu$ then $A' \in \xi$. But we have $A \cap secA' = \emptyset$ since for every $a \in A$, $a \wedge a' = 0$ and $a' \in A'$ so $a \notin secA'$ which is contradiction so $A \in \mu$.

(6) Similar to (5). And by Proposition 4, (7) is clear.

(9) Let $A \notin \xi$ i.e. $A' \in \mu$ then we have $A' \cap secA = \emptyset$ i.e. $A' \cap stack(secA) = \emptyset$ so $secA \notin \gamma$.

Conversely, let $secA \notin \gamma$ i.e. there exists $B \in \mu$ s.t. $B \cap stack(secA) = \emptyset$ so $B \cap secA = \emptyset$ i.e. $\forall b \in B$, $\exists a \in A$ s.t. $a \wedge b = 0$ so $b \leq a'$ therefore $B \prec A'$ and by (BC1) it implies $A' \in \mu$ so $A \notin \xi$.

(10) Let $A \in \gamma$ i.e. $\forall B \in \mu$, $B \cap stackA \neq \emptyset$.

We consider $B = \{d' \in L | \forall a \in A, d \wedge a \neq 0\}$. For every $d' \in B$, there is not any $a \in A$ s.t. $a \leq d'$ i.e. for every $d' \in B$, $d' \notin stackA$ so $\{d' \in L | \forall a \in A, d \wedge a \neq 0\} \notin \mu$ therefore $\{d \in L | \forall a \in A, d \wedge a \neq 0\} \in \xi$ i.e. $secA \in \xi$.

Conversely, $secA \in \xi$ by (9), $sec^2 A \in \gamma$ i.e. $stackA \in \gamma$ and since $stackA << A$ by (BS1) it implies $A \in \gamma$.

(8) Let $A \in \mu$ and $B \in \gamma$ then by (7) we have $A \cap stackB \neq \emptyset$.

Conversely, let $\forall B \in \gamma$, $A \cap stackB \neq \emptyset$, if $A \notin \mu$ then $A' \in \xi$ and by (9) $secA' \in \gamma$ and by assumption $A \cap stack(secA') \neq \emptyset$ i.e. $A \cap secA' \neq \emptyset$ which is contradiction so $A \in \mu$.

(11) By (7) is clear. And (12) by (8) is clear.

**Proposition 6** If $(L, \xi)$ is a B-nearness frame and $\overline{\xi}$ and $\mu$ are respectively corresponding B-farness and B-covering then the following conditions are equivalent:

- $(C)$ If every finite corefinement of $A$ belongs to $\xi$ then $A$ belongs to $\xi$;
- $(C')$ If $A \in \overline{\xi}$ then there exists a finite corefinement $B$ of $A$ with $B \in \overline{\xi}$;
- $(C'')$ If $A \in \mu$ then there exists a finite refinements $B$ of $A$ with $B \in \mu$.

**Proof:** $(C) \Leftrightarrow (C')$: Obviously.

$(C') \Rightarrow (C'')$ : Let $A \in \mu$ then $A' \in \overline{\xi}$ so by $(C')$ there exists a finite $B' \in \overline{\xi}$ s.t. $B' << A'$ i.e. $B \prec A$ and since $B'$ is finite B-farness so $B$ is finite B-covering. So there exists a finite refinements $B$ of $A$ with $B \in \mu$.

$(C') \Leftarrow (C'')$ : Let $A \in \overline{\xi}$ so $A' \in \mu$ and by $(C'')$ there exists a finite $B' \in \mu$ s.t. $B' \prec A'$ so $B << A$ and since $B'$ is finite B-covering so $B$ is finite B-farness i.e. there exists finite corefinement of $A$ belongs to $\overline{\xi}$.

**Definition 9** [2] A B-nearness frame is called *contigual* iff it satisfies the condition $(C)$.

**Theorem 2** Let $(L, \xi)$ be a B-nearness frame then $\xi_c = \{A \subset L | \forall B << A, (B \text{ finite } \Rightarrow B \in \xi)\}$ is the smallest contigual B-nearness structure on $L$ contains $\xi$ that we call it contigual B-nearness structure on $L$ generated by $\xi$. In addition $\xi_f = (\xi_c)_f$ where $\xi_f = \{A \in \xi | A \text{ finite}\}$.

**Proof:** First we show $\xi_c$ is a B-nearness on $L$.

To (BN1): Let $A_1 << A_2$ and $A_2 \in \xi_c$. We have $\forall B << A_1$, $B << A_2$ so if $B$ is finite it implies that $B \in \xi$ which means $A_1 \in \xi_c$.

To (BN2): Let $\bigwedge A \neq 0$ and $B << A$ so $\bigwedge B \neq 0$ therefore $B \in \xi$ so $A \in \xi_c$.

To (BN3): By (BN2), $\emptyset \in \xi_c$ so $\xi_c \neq \emptyset$. And $\{0\} \notin \xi_c$ so $\xi_c \neq PL$.

To (BN4): Let $A \bigvee B \in \xi_c$ if $A \notin \xi_c$ and $B \notin \xi_c$ then $\exists\ C << A$ and $C$ is finite but $C \notin \xi$ and $\exists\ D << B$ and $D$ is finite but $D \notin \xi$ so $C \bigvee D \notin \xi$ and it is finite. But since $C << A$ and $D << B$ we have $\forall\ c_i \in C\ \exists\ a_i \in A$ s.t. $a_i \leq c_i$ and $\forall\ d_j \in D\ \exists\ b_j \in B$ s.t. $b_j \leq d_j$ therefore $\forall c_i \vee d_j \in C \bigvee D\ \exists\ a_i \vee b_j \in A \bigvee B$ s.t. $a_i \vee b_j \leq c_i \vee d_j$ i.e. $C \bigvee D << A \bigvee B$ and since $C \bigvee D \notin \xi$ and it is finite it is contradiction to $A \bigvee B \in \xi_c$.

To (BN5′): Let $x \in L$, then $T = \{y \in L | x \leq y, \{x, y'\} \notin \xi\}$ and
$S = \{y \in L | x \leq y, \{x, y'\} \notin \xi_c\}$. Let $y \in T$ so $x \leq y$ and $\{x, y'\} \notin \xi$ if $y \notin S$ so $\{x, y'\} \in \xi_c$ by definition of $\xi_c$, $\{x, y'\} \in \xi$ that is contradiction. So we have $T \subseteq S$ therefore $\bigwedge S \leq \bigwedge T$ and we know $\bigwedge T = x$ also by definition of $S$, $x$ is its lower bound so $x = \bigwedge S$.

Therefore $\xi_c$ is a B-nearness on $L$.

Now let every finite corefinement of $A$ belongs to $\xi_c$ if $A \notin \xi_c$ then $\exists B << A$ where $B$ is finite and $B \notin \xi$ so $B \notin \xi_c$ that is contradiction so $\xi_c$ is contigual.

Now we show that $\xi_c$ is the smallest contigual B-nearness contains $\xi$.

Let $A \in \xi$ so by (BN1) for every $B << A$, we have $B \in \xi$ therefore $A \in \xi_c$. i.e. $\xi \subset \xi_c$. Suppose $\eta$ be an arbitrary contigual B-nearness contains $\xi$ and $A \in \xi_c$, so $\forall B << A$, if $B$ is finite then $B \in \xi$ therefore $B \in \eta$ and since $\eta$ is contigual, $A \in \eta$ i.e. $\xi_c \subset \eta$ so $\xi_c$ is the smallest contigual B-nearness contains $\xi$.

And obviously, $\xi_f = (\xi_c)_f$.

**Proposition 7** If $(L, \xi)$ is a B-nearness frame and $\overline{\xi}$, $\mu$ and $\gamma$ are respectively corresponding B-farness, B-covering and B-smallness then the following conditions are equivalent:

- $(U)$ If $A \in \overline{\xi}$ then there exists $B \in \overline{\xi}$ such that $\left\{st(b, B)^d | b \in B\right\} << A$;
- $(U')$ If $A \in \mu$ then there exists $B \in \mu$ such that $\{st(b, B) | b \in B\} \prec A$;
- $(U'')$ If $A \notin \gamma$ then $\exists B \subset L$ s.t. $secB \notin \gamma$ and $\left\{st(b, B)^d | b \in B\right\} << secA$.

**Proof:**

($(U) \Rightarrow (U')$) Let $A \in \mu$ i.e. $A' \in \overline{\xi}$ then by $(U)$, there exists $B' \in \overline{\xi}$ s.t. $\left\{st(b', B')^d | b' \in B'\right\} << A'$ i.e. for every $b' \in B'$ there exists $a' \in A'$ s.t. $a' \leq st(b', B')^d$ i.e. $a' \leq \bigwedge\{c' \in B' | c' \vee b' \neq 1\}$ so $\bigvee\{c \in B | c \wedge b \neq 0\} \leq a$. Therefore for every $b \in B$ there exists $a \in A$ s.t. $st(b, B) \leq a$ i.e. $\{st(b, B) | b \in B\} \prec A$.

($(U) \Leftarrow (U')$) Let $A \in \overline{\xi}$ i.e. $A' \in \mu$ then by $(U')$ there exists $B' \in \mu$ such that $\{st(b', B') | b' \in B'\} \prec A'$ i.e. for every $b' \in B'$ there exists $a' \in A'$ s.t. $st(b', B') \leq a'$ i.e. $\bigvee\{c' \in B' | c' \wedge b' \neq 0\} \leq a'$ so $a \leq \bigwedge\{c \in B | c \vee b \neq 1\}$. Therefore for every $b \in B$, there exists $a \in A$ s.t. $a \leq st(b, B)^d$ i.e. $\left\{st(b, B)^d | b \in B\right\} << A$.

($(U) \Rightarrow (U'')$) If $A \notin \gamma$ then $secA \in \overline{\xi}$ so by $(U)$ there exists $B \in \overline{\xi}$ such that $\{st(b,B)^d | b \in B\} << secA$ and $B \in \overline{\xi}$ implies $secB \notin \gamma$.

($(U) \Leftarrow (U'')$) If $A \in \overline{\xi}$ i.e. $secA \notin \gamma$ then by $(U'')$ $\exists B \subset L$ s.t. $secB \notin \gamma$ and $\{st(b,B)^d | b \in B\} << sec^2 A = stackA$ and we know $stackA << A$ so $\{st(b,B)^d | b \in B\} << A$ also $secB \notin \gamma$ implies $B \in \overline{\xi}$.

**Definition 10** [2] B-nearness frame $(L, \xi)$ is called *uniform* iff it satisfies to condition $(U)$.

We denote by **UBCFrm** the category of uniform B-covering frames and B-covering homomorphisms.

Also we denote by **UBNFrm** the category of uniform B-nearness frames and B-nearness homomorphisms.

## 5 Relation between B-Nearness on frames and subframes

Let $L$ be a Boolean frame and $a \in L$ then $\downarrow a = \{x \in L | x \leq a\}$ is Boolean frame with $\wedge$ and $\vee$ defined as in $L$. The top of $\downarrow a$ is $a$ and the bottom of $\downarrow a$ is the bottom of $L$.

Let $(L, \xi)$ be a B-nearness frame and $\overline{\xi}$ its corresponding farness. Let $a \in L$. For each $A \subseteq L$

$$a \wedge A = \{a \wedge x | x \in A\}$$

and

$$a \wedge \xi = \{A \in P(\downarrow a) | A \in \xi\} \quad \text{and} \quad a \wedge \overline{\xi} = \left\{A \in P(\downarrow a) | A \in \overline{\xi}\right\}$$

**Theorem 3** If $(L, \xi)$ is a B-nearness frame and $a \in L$ then $a \wedge \xi$ is a nearness on $\downarrow a$ and $a \wedge \overline{\xi}$ is corresponding B-farness on $\downarrow a$.

**Proof:** (BN1) to (BN4) are obvious. We prove only (BN5).

To (BN5): Let $x \in \downarrow a$ and

$$S = \left\{y \in L | x \leq st(y, A)^d \text{ for some } A \notin \xi\right\} \text{ and}$$
$$T = \left\{w \in \downarrow a | x \leq st(w, B)^d \text{ for some } B \notin a \wedge \xi\right\}$$

Let $z \in S$ by (BN2) and proposition 2, $x \leq z$ and since $x \leq a$ so $x \leq a \wedge z$. Choose $A \notin \xi$ s.t. $x \leq st(z, A)^d$, since $A \ll a \wedge A$ so by (BN1), $a \wedge A \notin \xi$ and by definition, $a \wedge A \notin a \wedge \xi$

$$\begin{aligned} st((a \wedge z), (a \wedge A))^d &= \bigwedge \{a \wedge h | h \in A, (a \wedge z) \vee (a \wedge h) \neq a\} \\ &= \bigwedge \{a \wedge h | h \in A, a \wedge (z \vee h) \neq a\} \\ &= a \wedge (\bigwedge \{h \in A | a \nleq z \vee h\} \end{aligned}$$

Obviously $\{h \in A | a \nleq z \vee h\} \subset \{h \in A | z \vee h \neq 1\}$ so $\bigwedge \{h \in A | z \vee h \neq 1\} \leq \bigwedge \{h \in A | a \nleq z \vee h\}$ and we have $x \leq st(z, A)^d = \bigwedge \{h \in A | z \vee h \neq 1\}$ so $x \leq \bigwedge \{h \in A | a \nleq z \vee h\}$ and since $x \leq a$ therefore $x \leq a \wedge (\bigwedge \{h \in A | a \nleq z \vee h\}$ i.e. $x \leq st((a \wedge z), (a \wedge A))^d$ so $a \wedge z \in T$ therefore $a \wedge S \subseteq T$ so $\bigwedge T \leq \bigwedge (a \wedge S) \leq \bigwedge S$ since $\xi$ is B-nearness so $x = \bigwedge S$ so $\bigwedge T \leq x$ and for any $w \in T$ by (BN2) and proposition 2, we have $x \leq w$ i.e. $x$ is a lower bound for $T$ so $\bigwedge T = x$.

So $a \wedge \xi$ is a B-nearness on $\downarrow a$ generated by $\xi$.

Now we show that $a \wedge \overline{\xi}$ is its corresponding B-farness on $\downarrow a$

$$\overline{a \wedge \xi} = \{A \subset P(\downarrow a) | A \notin a \wedge \xi\} \text{ i.e. } \overline{a \wedge \xi} = \{A \subset P(\downarrow a) | A \notin \xi\} \text{ i.e.}$$
$$\overline{a \wedge \xi} = \left\{A \subset P(\downarrow a) | A \in \overline{\xi}\right\} = a \wedge \overline{\xi}.$$

**Proposition 8** Let $a \leq b$ in $L$ and $\overline{\xi}_b$ and $\overline{\xi}_a$ are respectively B-farnesses on $\downarrow b$ and $\downarrow a$ generated by an unknown B-farness on $L$ then $\overline{\xi}_b$ and $\overline{\xi}_a$ satisfy on the following relations :

(i) If $D \in \overline{\xi}_b$ then $D << C$ for some $C \in \overline{\xi}_a$.

(ii) If $C \in \overline{\xi}_a$ and $C << D$ for some $D \in P(\downarrow b)$ then $D \in \overline{\xi}_b$

**Proof:** Obviously.

**Definition 11** B-nearness frame $(L, \xi)$ is called *graded* iff $\{A, secA\} \subseteq \xi$ implies $\xi(A) \in \xi$, where $\xi(A) = \{x \in L | (\{x\} \cup A) \in \xi\}$.

**Proposition 9** Let $(L, \xi)$ be a B-nearness frame and $a \in L$ then following result holds

(i) If $(L, \xi)$ is graded then $a \wedge \xi$ is also a graded B-nearness on $\downarrow a$.

(ii) If $(L, \xi)$ is contigual then $a \wedge \xi$ is also a contigual B-nearness on $\downarrow a$.

**Proof:** (i) Let $\{A, sec_{\downarrow a} A\} \subset a \wedge \xi$ since $A \in P(\downarrow a)$, $sec_{\downarrow a} A = a \wedge sec_L A$ and $sec_L A << a \wedge sec_L A$ so $\{A, sec_L A\} \subset \xi$ then $\xi(A) \in \xi$ but $(a \wedge \xi)(A) \subseteq \xi(A)$ so $(a \wedge \xi)(A) \in \xi$ and $(a \wedge \xi)(A) \subset P(\downarrow a)$ so $(a \wedge \xi)(A) \in a \wedge \xi$.

(ii) Let $A \subseteq \downarrow a$ and every finite corefinemen of $A$ in $\downarrow a$ belongs to $a \wedge \xi$ . If there exists $B$ finite subset of $L$ s.t. $B << A$ and $B \notin \xi$, since $B << a \wedge B$ then by (BN1), $a \wedge B \notin \xi$ but since $A \in \downarrow a$ and $B << A$ so $\forall\, a \wedge b \in a \wedge B\ \exists\, x \in A$ s.t. $x \leq a \wedge b$ i.e. $a \wedge B << A$ then by assumption $a \wedge B \in a \wedge \xi$ and so $a \wedge B \in \xi$ that is contradiction so every finite corefinemen of $A$ in $L$ belongs to $\xi$, therefore $A \in \xi$ and so $A \in a \wedge \xi$.

Let $L$ be a Boolean frame and $a \in L$ then $\uparrow a = \{x \in L | a \leq x\}$ is Boolean frame with $\wedge$ and $\vee$ defined as in $L$. The top of $\uparrow a$ is the top of $L$ and the bottom of $\uparrow a$ is $a$.

Let $(L, \xi)$ be a nearness frame and $\overline{\xi}$ its corresponding B-farness. Let $a \in L$.

For each $A \subseteq L$

$$a \vee A = \{a \vee x | x \in A\}$$

and

$$a \vee \overline{\xi} = \left\{B \in P(\uparrow a) | a \vee A << B, \text{ for some } A \in \overline{\xi}\right\}$$

and $a \vee \xi$ is its corresponding B-nearness.

**Theorem 4** If $\overline{\xi}$ is a B-farness on $L$ and $a \in L$ then $a \vee \overline{\xi}$ is a B-farness on $\uparrow a$.

**Proof:**

To (BF1): Let $C << D$ and $C \in a \vee \overline{\xi}$ so $\exists A \in \overline{\xi}$ s.t. $a \vee A << C$ and since $C << D$ so $a \vee A << D$ i.e. $D \in a \vee \overline{\xi}$

To (BF2): Let $C \in a \vee \overline{\xi}$. If $\bigwedge C \neq a$ then $\exists\ z > a$ s.t. $\bigwedge C = z$ but since $C \in a \vee \overline{\xi}$ so $\exists A \in \overline{\xi}$ s.t. $a \vee A << C$ i.e. $\forall a \vee x \in a \vee A\ \exists c \in C$ s.t. $c \leq a \vee x$ so $\forall a \vee x \in a \vee A$, $z \leq a \vee x$ i.e. $z \leq \bigwedge(a \vee A) = a \vee (\bigwedge A)$ but $\bigwedge A = 0$ therefore $z \leq a$ that is contradiction to assumption. so $\bigwedge C = a$.

To (BF3): For every $A \in \overline{\xi}$, $a \vee A << \{a\}$ i.e. $\{a\} \in a \vee \overline{\xi}$ so $a \vee \overline{\xi} \neq \emptyset$. Now let $b > a$ by (BF2) we know $\{b\} \notin a \vee \overline{\xi}$ so $a \vee \overline{\xi} \neq P(\uparrow a)$.

To (BF4): Let $C \in a \vee \overline{\xi}$ and $D \in a \vee \overline{\xi}$ so $\exists\ A \in \overline{\xi}$ s.t. $a \vee A << C$ and $\exists\ B \in \overline{\xi}$ s.t. $a \vee B << D$ so $A \bigvee B \in \overline{\xi}$. Now let $a \vee (x \vee y) \in a \vee (A \bigvee B)$ where $x \in A$ and $y \in B$ i.e. $(a \vee x) \vee (a \vee y) \in a \vee (A \bigvee B)$ since $a \vee x \in a \vee A$ and $a \vee A << C$, $\exists\ c \in C$ s.t. $c \leq a \vee x$ similarly $\exists\ d \in D$ s.t. $d \leq a \vee y$ therefore $c \vee d \leq (a \vee x) \vee (a \vee y) = a \vee (x \vee y)$ i.e. $a \vee (A \bigvee B) << C \bigvee D$. and since $C \bigvee D \in P(\uparrow a)$ then $C \bigvee D \in a \vee \overline{\xi}$.

To (BF5): Let $x \in \uparrow a$ and

$$S = \left\{y \in L | x \leq st(y, A)^d \text{ for some } A \in \overline{\xi}\right\} \text{ and}$$
$$T = \left\{w \in \uparrow a | x \leq st(w, B)^d \text{ for some } B \in a \vee \overline{\xi}\right\}$$

Let $z \in S$ then by (BF2) and proposition 2, $x \leq z$ and $a \leq x$ so $z \in \uparrow a$. Now we choose $A \in \overline{\xi}$ s.t. $x \leq st(z, A)^d$. Obviously, $a \vee A \in a \vee \overline{\xi}$.

$$\begin{aligned}
st(z, (a \vee A))^d &= \bigwedge \{a \vee h | h \in A \text{ and } z \vee (a \vee h) \neq 1\} \\
&= a \vee (\bigwedge \{h \in A | z \vee h \neq 1\}) \\
&\geq \bigwedge \{h \in A | z \vee h \neq 1\} \\
&= st(z, A)^d \geq x
\end{aligned}$$

So $z \in T$ i.e. $S \subseteq T$ therefore $\bigwedge T \leq \bigwedge S$ and since $\bigwedge S = x$ and similar to proposition 2, $x$ is a lower bound for every $w \in T$ so we have $\bigwedge T = x$.

so $a \vee \overline{\xi}$ is a B-farness on $\uparrow a$ generated by $\overline{\xi}$.

**Proposition 10** Let $(L, \xi)$ be a B-nearness frame and $a \in L$ then following result holds

(i) If $(L, \xi)$ is uniform then $a \vee \xi$ is also a uniform B-nearness on $\uparrow a$;

(ii) If $(L, \xi)$ is contigual then $a \vee \xi$ is also a contigual B-nearness on $\uparrow a$.

**Proof:** (i) Let $C \in a \vee \overline{\xi}$ then there exists $A \in \overline{\xi}$ such that $a \vee A << C$ and since $\overline{\xi}$ is uniform so $\exists B \in \overline{\xi}$ s.t. $\left\{st(b, B)^d | b \in B\right\} << A$ i.e. $\forall b \in B$, $\exists x \in A$ s.t.
$x \leq \bigwedge \{b_i \in B | b_i \vee b \neq 1\}$ therefore we have $\forall a \vee b \in a \vee B (\in a \vee \overline{\xi})$, $\exists a \vee x \in a \vee A$ s.t. $a \vee x \leq \bigwedge \{a \vee b_i \in a \vee B | b_i \vee b \neq 1\}$.

But $\{a \vee b_i \in a \vee B | b_i \vee b \vee a \neq 1\} \subseteq \{a \vee b_i \in a \vee B | b_i \vee b \neq 1\}$ so
$\bigwedge \{a \vee b_i \in a \vee B | b_i \vee b \neq 1\} \leq \bigwedge \{a \vee b_i \in a \vee B | (a \vee b_i) \vee (a \vee b) \neq 1\}$
then we have $a \vee x \leq \bigwedge \{a \vee b_i \in a \vee B | (a \vee b_i) \vee (a \vee b) \neq 1\}$ i.e.
$\left\{st(a \vee b, a \vee B)^d | a \vee b \in a \vee B\right\} << a \vee A$ and since $a \vee A << C$ so
$\left\{st(a \vee b, a \vee B)^d | a \vee b \in a \vee B\right\} << C$ i.e. $a \vee \xi$ is uniform.

(ii) Let $C \in a \vee \overline{\xi}$ then there exists $A \in \overline{\xi}$ s.t. $a \vee A << C$. But $\overline{\xi}$ is contigual so there exists finite $B \in \overline{\xi}$ s.t. $B << A$ so $a \vee B$ is finite and it belongs to $a \vee \overline{\xi}$ and also $a \vee B << a \vee A$ so $a \vee B << C$ i.e. $a \vee \xi$ is contigual.

## 6 B-Nearness frame and Complete Near Space

**Definition 12** Let $(L, \xi)$ be a B-nearness frame. A nonempty subset $A$ of $L$ is called $\xi$-*cluster* iff $A$ is a maximal element of the set $\xi$ ordered by set inclusion.

**Theorem 5** Let $(L, \xi)$ be a B-nearness frame, $X^*$ be set of all $\xi$-clusters and
$\xi^* = \{\Omega \subset PX^* | \bigcup \{\bigcap \omega | \omega \in \Omega\} \in \xi\}$ then $(X^*, \xi^*)$ is complete nearness space induced by B-nearness frame $(L, \xi)$.

**Proof:** We prove that $(X^*, \xi^*)$ satisfies in all conditions of nearness space.

To $(N1)$ i.e. Let $\Omega_1 << \Omega_2$ and $\Omega_2 \in \xi^*$ then $\Omega_1 \in \xi^*$.

If $\Omega_1 \notin \xi^*$ then $\bigcup \{\bigcap \omega | \omega \in \Omega_1\} \notin \xi$ therefore $\forall A_i \in X^*$, $\bigcup \{\bigcap \omega | \omega \in \Omega_1\} \nsubseteq A_i$ i.e. $\forall A_i \in X^*$, $\exists x_i \in \bigcup \{\bigcap \omega | \omega \in \Omega_1\}$ s.t. $x_i \notin A_i$ but for some $\omega \in \Omega_1$, $x_i \in \bigcap \omega$ say $x_i \in \bigcap \omega_i$ when $\omega_i \in \Omega_1$.

Since $\Omega_1 << \Omega_2$, $\forall \omega_1 \in \Omega_1$, $\exists \omega_2 \in \Omega_2$ s.t. $\omega_2 \subseteq \omega_1$ we have $\bigcap \omega_1 \subseteq \bigcap \omega_2$. And since $\forall A_i \in X^*$, $\exists x_i \in \bigcap \omega_i$ where $x_i \notin A_i$ so $\exists \omega'_i \in \Omega_2$ s.t. $\omega'_i \subseteq \omega_i$ so $\bigcap \omega_i \subseteq \bigcap \omega'_i$ therefore $x_i \in \bigcap \omega'_i$ and so $x_i \in \bigcup \{\bigcap \omega | \omega \in \Omega_2\}$ where $x_i \notin A_i$ i.e. $\forall A_i \in X^*$, $\bigcup \{\bigcap \omega | \omega \in \Omega_2\} \nsubseteq A_i$ so $\bigcup \{\bigcap \omega | \omega \in \Omega_2\} \notin \xi$ i.e. $\Omega_2 \notin \xi^*$ that is contradiction so $\Omega_1 \in \xi^*$.

To $(N2)$ i.e. If $\bigcap \Omega \neq \emptyset$ then $\Omega \in \xi^*$.

Let $\bigcap\Omega \neq \emptyset$ so $\exists A \in X^*$ s.t. $\forall \omega \in \Omega$, $A \in \omega$ therefore
$\forall \omega \in \Omega$, $\bigcap\omega \subseteq A$ so $\bigcup\{\bigcap\omega|\omega \in \Omega\} \subseteq A$ and since $A \in \xi$ so $\bigcup\{\bigcap\omega|\omega \in \Omega\} \in \xi$ i.e. $\Omega \in \xi^*$.

To $(N3)$ i.e. $\emptyset \neq \xi^* \neq P^2X^*$.

Since $\xi \neq \emptyset$ so $\exists A \in X^*$ and by definition of $X^*$ obviously $\{\{A\}\} \in \xi^*$ so $\xi^* \neq \emptyset$. And if $\{\emptyset\} \in \xi^*$ then $\bigcup\{\bigcap\emptyset\}\,(= L) \in \xi$ that is contradiction to $\xi \neq PL$ so $\xi^* \neq P^2X^*$.

To $(N4)$ i.e. If $\Omega_1 \bigvee \Omega_2 \in \xi^*$ then $\Omega_1 \in \xi^*$ or $\Omega_2 \in \xi^*$,
where $\Omega_1 \bigvee \Omega_2 = \{\omega_1 \cup \omega_2 | \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}$.

Let $\Omega_1 \bigvee \Omega_2 \in \xi^*$. If $\Omega_1 \notin \xi^*$ and $\Omega_2 \notin \xi^*$ then
$\Omega_1 \notin \xi^* \Rightarrow \bigcup\{\bigcap\omega_i|\omega_i \in \Omega_1\} \notin \xi$, we call $U_1 = \bigcup\{\bigcap\omega_i|\omega_i \in \Omega_1\}$ and
$\Omega_2 \notin \xi^* \Rightarrow \bigcup\{\bigcap\omega_j|\omega_j \in \Omega_2\} \notin \xi$, we call $U_2 = \bigcup\{\bigcap\omega_j|\omega_j \in \Omega_2\}$.

Since $U_1 \notin \xi$ and $U_2 \notin \xi$ then $U_1 \bigvee U_2 \notin \xi$ i.e.

$$\{x \vee y | x \in U_1, y \in U_2\} \notin \xi \tag{I}$$

Let $x \in U_1$ then $\exists \omega_1 \in \Omega_1$ s.t. $x \in \bigcap\omega_1$ therefore $\forall A_i \in \omega_1$, $x \in A_i$ so $\forall A_i \in \omega_1$, $A_i \bigcup \{x \vee z\} << A_i$ where $z \in L$ so $A_i \bigcup \{x \vee z\} \in \xi$ but since each $A_i$ is $\xi$-cluster so $\forall A_i \in \omega_1$, $x \vee z \in A_i$ for any $z \in L$.

Similarly Let $y \in U_2$ then $\exists \omega_2 \in \Omega_2$ s.t. $y \in \bigcap\omega_2$ therefore $\forall B_j \in \omega_2$, $y \in B_j$ so $\forall B_j \in \omega_2$, $y \vee z \in B_j$ for any $z \in L$. So we have
$\forall A_i \in \omega_1$ and $\forall B_j \in \omega_2$, $x \vee y \in A_i$ and $x \vee y \in B_j$ i.e. $x \vee y \in \bigcap(\omega_1 \cup \omega_2)$ so $x \vee y \in \bigcup\{\bigcap(\omega_i \cup \omega_j)|\omega_i \in \Omega_1, \omega_j \in \Omega_2\}$ therefore
$\{x \vee y | x \in U_1, y \in U_2\} \subseteq \bigcup\{\bigcap(\omega_i \cup \omega_j)|\omega_i \in \Omega_1, \omega_j \in \Omega_2\}$
and by (I) we have $\bigcup\{\bigcap(\omega_i \cup \omega_j)|\omega_i \in \Omega_1, \omega_j \in \Omega_2\} \notin \xi$ i.e. $\Omega_1 \bigvee \Omega_2 \notin \xi^*$ that is contradiction so $\Omega_1 \in \xi^*$ or $\Omega_2 \in \xi^*$.

To $(N5)$ i.e. If $\{cl^*_\xi\omega|\omega \in \Omega\} \in \xi^*$ then $\Omega \in \xi^*$ where $cl^*_\xi\omega = \{A \in X^* | \{\omega, \{A\}\} \in \xi^*\}$.

Let $\omega \subseteq X^*$ so $cl^*_\xi\omega = \{A \in X^* | ((\bigcap\omega) \cup A) \in \xi\}$ but since $A$ is $\xi$-cluster so it is maximal in $\xi$ then $cl^*_\xi\omega = \{A \in X^* | \bigcap\omega \subseteq A\}$ therefore $\forall \omega \subseteq X^*$, $\bigcap\omega \subseteq \bigcap cl^*_\xi\omega$.
Now let $\{cl^*_\xi\omega|\omega \in \Omega\} \in \xi^*$ so $\bigcup\{\bigcap(cl^*_\xi\omega)|\omega \in \Omega\} \in \xi$ and we know
$\bigcap\omega \subseteq \bigcap cl^*_\xi\omega$ so $\bigcup\{\bigcap\omega|\omega \in \Omega\} \subseteq \bigcup\{\bigcap(cl^*_\xi\omega)|\omega \in \Omega\}$ so
$\bigcup\{\bigcap\omega|\omega \in \Omega\} \in \xi$ i.e. $\Omega \in \xi^*$.
Therefore $(X^*, \xi^*)$ is a nearness space.

Now we have to show that $(X^*, \xi^*)$ is complete.

Let $\Omega \in \xi^*$ be a $\xi^*$-*cluster* by definition $\bigcup\{\bigcap\omega|\omega \in \Omega\} \in \xi$ so there exists $\xi$-*cluster*, A, s.t. $\bigcup\{\bigcap\omega|\omega \in \Omega\} \subset A$. We consider $\Omega' = \Omega \cup \{\{A\}\}$ obviously $\bigcup\{\bigcap\omega'|\omega' \in \Omega'\} \subset A$ therefore $\Omega' \in \xi^*$, since $\Omega$ is $\xi^*$-*cluster* so $\{A\} \in \Omega$ therefore $(X^*, \xi^*)$ is a complete nearness space.

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# Local antimaximum principle for the Schrödinger operator in $\mathbb{R}^N$

ABSTRACT. We consider in this paper equations defined in $\mathbb{R}^N$ involving Schrödinger operators with indefinite weight functions and with potentials which tend to infinity at infinity. After recalling the existence of principal eigenvalues and the maximum principle, we study the local antimaximum principle.

KEY WORDS. Schrödinger operator, indefinite weight, antimaximum principle

## 1 Introduction

We consider in this paper the Schrödinger operator $-\Delta + q$ defined on $\mathbb{R}^N$ associated with the indefinite weight $m$ where $q$ is a potential which satisfies the following hypothesis:

$(\mathbf{H_q^1})$ $q \in L^2_{loc}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N) \cap L^p_{loc}(\mathbb{R}^N)$, $p > \frac{N}{2}$, such that $\lim_{|x|\to\infty} q(x) = \infty$ and $q \geq cst > 0$.

and where the weight $m$ satisfies one of the following hypotheses:

$(\mathbf{H_m'^1})$ $m \in L^\infty(\mathbb{R}^N)$, $m$ is positive in the open subset $\Omega_m^+ = \{x \in \mathbb{R}^N, m(x) > 0\}$ with non zero measure and $m$ is negative in the open subset $\Omega_m^- = \{x \in \mathbb{R}^N, m(x) < 0\}$ with non zero measure.

$(\mathbf{H_m'^2})$ **(i)** $m \in L^{N/2}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$ $(N \geq 3)$, $meas(\Omega_m^+) > 0$, $meas(\Omega_m^-) > 0$.

**(ii)** $m = m_1 - m_2$, $m_1 \geq 0$, $m_1 \in L^\infty(\mathbb{R}^N)$, $m_2 \geq 0$, $m_2 \in L^\infty_{loc}(\mathbb{R}^N)$.

Mainly, this paper deals with the local antimaximum principle for the following equation

$$(-\Delta + q)u = \lambda m u + f \text{ in } \mathbb{R}^N, \tag{1.1}$$

where $\lambda$ is a real parameter and f satisfies the following hypothesis:

$(\mathbf{H_f^1})$ $f \in L^2(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$.

As in [2, 3] we introduce the quadratic form

$$(v,w)_q = \int_{\mathbb{R}^N} \nabla v \cdot \nabla w + qvw$$

defined for every pair

$$v, w \in V_q(\mathbb{R}^N) := \{f \in L^2(\mathbb{R}^N,\ (f,f)_q < \infty\}.$$

Notice that $V_q(\mathbb{R}^N)$ is a Hilbert space with the inner product $(v,w)_q$ and the norm

$$\|v\|_q = ((v,v)_q)^{1/2} = \left(\int_{\mathbb{R}^N} [|\nabla v|^2 + qv^2]\right)^{1/2}.$$

The set $D(\mathbb{R}^N)$, which is the set of $C^\infty$ functions with compact supports, is a dense linear subspace of $V_q(\mathbb{R}^N)$. By the Lax-Milgram theorem, the Schrödinger operator $L = -\Delta + q$ in $L^2(\mathbb{R}^N)$ is defined to be the selfadjoint operator in $L^2(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N} (Lv)w = (v,w)_q \text{ for all } v, w \in D(\mathbb{R}^N).$$

We denote by $D(L)$ its domain (strong domain) and $V_q(\mathbb{R}^N)$ is its weak domain. In the following a function $u \in V_q(\mathbb{R}^N)$ will be called a solution of (1.1) if it is a weak solution of (1.1) i.e. if $\int_{\mathbb{R}^N} \nabla u \cdot \nabla \phi + qu\phi = \lambda \int_{\mathbb{R}^N} mu\phi + \int_{\mathbb{R}^N} f\phi$ for all $\phi \in D(\mathbb{R}^N)$. We recall that the embedding of $V_q(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$ is compact.

We add another hypothesis upon the potential $q$ which assures that any element of the weak domain of the operator $L = -\Delta + q$ belongs to the strong domain $D(L)$. It is the following hypothesis.

$(\mathbf{H_q^2})$ There exists a positive constant $C$ such that for all $x \in \mathbb{R}^N$ and all $h \in \mathbb{R}^N$, $h \neq 0$, $|\frac{q(x+h)-q(x)}{|h|}| \leq C\ \sqrt{q(x)}$.

Note that for example, the potential $q(x) = 1 + |x|$ satisfies $(\mathbf{H_q^2})$. And we recall the following proposition in [7], based on the methods of translations due to Nirenberg.

**Proposition 1.1** *Assume that the potential $q$ satisfy $(\mathbf{H_q^1})$ and $(\mathbf{H_q^2})$. Let $u$ be a weak solution of $(-\Delta + q)u = f$ in $\mathbb{R}^N$ with $f \in L^2(\mathbb{R}^N)$. Then $u \in H^2(\mathbb{R}^N)$, $qu \in L^2(\mathbb{R}^N)$ and therefore $u \in D(L)$.*

Our assumptions on the weight $m$ guarantee the existence of a unique principal and positive eigenvalue $\lambda_{1,q,m} > 0$ associated with a positive eigenfunction $\phi_{1,q,m} > 0$, and also the existence and uniqueness of a principal negative eigenvalue $\tilde{\lambda}_{1,q,m} < 0$ associated with a positive eigenfunction $\tilde{\phi}_{1,q,m} > 0$ (see [6]). We also recall a variational characterization of these eigenvalues and that will be essential for the proof of the local antimaximum principle. The problem of the existence of principal eigenvalues has been studied for the Laplacian and the p-Laplacian operators associated with a weight, in bounded domains (see for example [14]), in $\mathbb{R}^N$ (see for example [5]), for the Schrödinger operator $-\Delta + q$ associated with a weight $m$ in $\mathbb{R}^N$ (see [6, 7]). We also recall the maximum principle for (1.1): if $u$ is one weak solution of (1.1), if $f \geq 0$ and if $\tilde{\lambda}_{1,q,m} < \lambda < \lambda_{1,q,m}$, then $u \geq 0$. Note that the maximum principle has been extensively studied for equations or systems (see for example [8, 11–13, 21, 23, 25, 26]).

Afterwards we study the local antimaximum principle: we denote by $B_R$ the open ball in $\mathbb{R}^N$ of center 0 and radius $R$ ; if $f \geq 0$, $f \not\equiv 0$, then there exists a constant $\delta = \delta(f, R) > 0$ such that for all $\lambda \in ]\lambda_{1,q,m}, \lambda_{1,q,m} + \delta[$, any solution $u$ of (1.1) is negative in $\overline{B}_R$.

In various common versions of the antimaximum principle in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, besides the assumption $f \geq 0$, $f \not\equiv 0$ in $\Omega$, it is only assumed that $f \in L^p(\Omega)$ for some $p > N$ (cf [10, Theorem 1 p.222], [25, 26]). The case of the Schrödinger operator on $\mathbb{R}^N$ is more difficult; the hypothesis $f \in L^p(\Omega)$ $(p > N)$ is no longer sufficient (see [2, 3] for the Schrödinger operator with no weight and [7] for the Schrödinger operator with a positive bounded weight $m$). Therefore the two main difficulties here are the unboundedness of the domain $\mathbb{R}^N$ and the weight $m$ which is not a positive bounded function.

We do not use the ideas expressed in [2, 3, 10, 20, 25, 26] where the antimaximum principle is obtained by a decomposition of the resolvant of the operator near the principal and positive eigenvalue and by projecting on the eigenspace generated by the eigenfunction associated with this eigenvalue. Indeed because of the unboundedness of our domain, we cannot proceed as for example in Hess (see [20]) where the antimaximum principle is studied for the Laplacian operator with an indefinite weight function but in a bounded domain. And furthermore because of our weight which is an indefinite function, we cannot proceed as in [2, 3, 7] where the antimaximum principle is studied for the Schrödinger operator $-\Delta + q$ on $\mathbb{R}^N$ with a positive bounded weight $m$ ($m = 1$ in [2, 3]): more precisely in these former papers $(\int_{\mathbb{R}^N} mu^2)^{1/2}$ must be a norm, equivalent to the usual norm in $L^2(\mathbb{R}^N)$.

Thus for the proof of the local antimaximum principle, we follow a method developed in [15, 24]. This method has been first established for the Laplacian operator in $\mathbb{R}^N$, $N \geq 3$, and $f \in L^\infty(\mathbb{R}^N)$ and for the p-Laplacian operator with a nonpositive weight $m$ at infinity (see [24]), then it has been extended to the p-Laplacian operator in $\mathbb{R}^N$ in [15]. This method is based on a nonexistence result of nonnegative solutions for (1.1) if $\lambda > \lambda_{1,q,m}$ (see Proposition 3.1) and also on estimates given by the regularity $C^1$ of any solution $u$ of (1.1). We can get this regularity either by using a regularity result of Tolksdorf in [27] and Serrin $L^\infty(B_R)$ estimates for $u$ (see [22]) as in [15, 24] or more classically by the local $L^p$-regularity theory.

Indeed, first note that any solution $u$ of (1.1) is continuous (see [1, Theorem 0.1 p.3], [23, Theorem 7.1 p.232]). Moreover if $u \in D(L)$ and $(-\Delta+q)u = f \in L^2(\mathbb{R}^N)$ with $f \in L^p_{loc}(\mathbb{R}^N)$ for some $p$ with $2 \le p < \infty$ then the local $L^p$-regularity theory yields $u \in W^{2,p}_{loc}(\mathbb{R}^N)$ (see [17, Theorem 9.15 p.241]). In particular, if $p > N$ then $u \in C^1(\mathbb{R}^N)$ by the Sobolev embedding theorem (see [17, Theorem 7.10 p. 155]).

Therefore these results for the Schrödinger operator $-\Delta + q$ associated with an indefinite weight $m$ extend here the results of the antimaximum principle for the Laplacien operator in a bounded domain (see [10, 20, 26]) and for the p-Laplacien operator in $\mathbb{R}^N$ (see [15, 24]). Note that extensions of maximum and antimaximum principles, respectively called ground state positivity and negativity (or also called fundamental positivity and negativity), are given for the Schrödinger operator in $\mathbb{R}^N$ without any weight (see [2, 4]) and with a positive weight (see [7]) but for a potential $q$ which is a perturbation of a radially symmetric potential and for a more restrictive set of functions $f$. Note in [15, Theorem 4.1] a result where the fundamental negativity in $\mathbb{R}^N$ is not verified for the Laplacien operator associated with an indefinite weight in dimension $N = 1$. Also recall examples given in [2, Example 2.1] and in [3, Example 4.1] where the global antimaximum principle in $\mathbb{R}^N$ is still not verified for the Schrödinger operator with no weight in dimension $N \ge 1$. Finally, we can cite among other papers the works of [16, 18, 19] where the antimaximum principle is studied either for the p-Laplacian operator or an elliptic operator of second order with a bounded weight on a bounded domain.

Our paper is organized as follows: In Section 2 we recall the existence of a principal positive (resp. negative) eigenvalue $\lambda_{1,q,m} > 0$ (resp. $\tilde{\lambda}_{1,q,m} < 0$) associated with a positive eigenfunction $\phi_{1,q,m} > 0$ (resp. $\tilde{\phi}_{1,q,m} > 0$). We also recall the classical maximum principle for (1.1) in the case of an indefinite weight $m$. In Section 3, we study the local antimaximum principle. Finally in Section 4, we extend the local antimaximum principle to the case of the system (4.1).

## 2 Existence of principal eigenvalues and maximum principle

First we recall in this section the existence of a unique positive principal eigenvalue $\lambda_{1,q,m}$ and of a unique negative principal eigenvalue $\tilde{\lambda}_{1,q,m}$ (see [6, Theorems 2.1,2.2,3.1]). So we assume in this paper that $q$ satisfies $(\mathbf{H^1_q})$, $(\mathbf{H^2_q})$ and $m$ satisfies $(\mathbf{H'^1_m})$ or $(\mathbf{H'^2_m})$.

**Theorem 2.1** *Assume that $q$ satisfies $(\mathbf{H^1_q})$, $(\mathbf{H^2_q})$ and $m$ satisfies $(\mathbf{H'^1_m})$ or $(\mathbf{H'^2_m})$(**i**). Then the operator $-\Delta + q$ associated with the weight $m$ has a unique positive principal eigenvalue $\lambda_{1,q,m}$ associated with a positive eigenfunction $\phi_{1,q,m} \in C^1(\mathbb{R}^N)$ normalized by*

$\int_{\mathbb{R}^N} m\phi_{1,q,m}^2 = 1$, $\lambda_{1,q,m}$ *is simple and* $(\lambda_{1,q,m}, \phi_{1,q,m})$ *satisfy*

$$(-\Delta + q)\phi_{1,q,m} = \lambda_{1,q,m}\ m\ \phi_{1,q,m} \text{ in } \mathbb{R}^N; \quad \lambda_{1,q,m} > 0; \quad \phi_{1,q,m} > 0.$$

$$\lambda_{1,q,m} = \inf\{\frac{\int_{\mathbb{R}^N}[|\nabla\phi|^2 + q\phi^2]}{\int_{\mathbb{R}^N} m\phi^2},\ \phi \in V_q(\mathbb{R}^N) \text{ s. t. } \int_{\mathbb{R}^N} m\phi^2 > 0\}, \tag{2.1}$$

*and this infimum is achieved for any function* $\phi = \alpha\phi_{1,q,m}$ *with* $\alpha \in \mathbb{R}$, $\alpha \neq 0$. *Moreover the operator* $-\Delta + q$ *associated with the weight* $m$ *has a unique negative principal eigenvalue* $\tilde{\lambda}_{1,q,m}$ *associated with a positive eigenfunction* $\tilde{\phi}_{1,q,m}$.

Now we recall the maximum principle for (1.1) (see ([6, Theorem 3.2])).

**Theorem 2.2** *Assume that* $q$ *satisfies* $(\mathbf{H_q^1})$, $(\mathbf{H_q^2})$ *and* $m$ *satisfies* $(\mathbf{H_m'^1})$ *or* $(\mathbf{H_m'^2})$(**i**). *Assume that* $f \in L^2(\mathbb{R}^N)$, $f \geq 0$ *and* $u$ *is a solution of* (1.1). *If* $\tilde{\lambda}_{1,q,m} < \lambda < \lambda_{1,q,m}$, *then* $u \geq 0$.

We conclude this section by adding the following proposition. We follow here [15, Proposition 2.1].

**Proposition 2.1** *Assume that* $q$ *satisfies* $(\mathbf{H_q^1})$, $(\mathbf{H_q^2})$ *and* $m$ *satisfies* $(\mathbf{H_m'^1})$ *or* $(\mathbf{H_m'^2})$(**i**)-(**ii**). *Then any minimizing sequence* $(u_k)$ *of* $\lambda_{1,q,m}$ *admits a subsequence which converges weakly in* $V_q(\mathbb{R}^N)$ *to some* $u$ *which realizes the infimum* (2.1); *and so there exists* $\alpha \in \mathbb{R}$, $\alpha \neq 0$ *such that* $u = \alpha\phi_{1,q,m}$.

**Proof:** First note that

$$\lambda_{1,q,m} = \inf\{\int_{\mathbb{R}^N}[|\nabla\phi|^2 + q\phi^2],\ \phi \in V_q(\mathbb{R}^N) \text{ s. t. } \int_{\mathbb{R}^N} m\phi^2 = 1\}. \tag{2.2}$$

Let now $(u_k)$ be a minimizing sequence. Then $(u_k)$ is a bounded sequence in $V_q(\mathbb{R}^N)$ and there exists $u \in V_q(\mathbb{R}^N)$ such that for a subsequence $(u_k)$ converges weakly to $u$ in $V_q(\mathbb{R}^N)$ (and strongly in $L^2(\mathbb{R}^N)$, and for a subsequence, still denoted by $(u_k)$, $u_k \to u$ a.e. in $\mathbb{R}^N$).

If $m$ satisfies $(\mathbf{H_m'^1})$, since the weight $m$ is bounded, by the Lebesgue dominated convergence theorem we get that $1 = \int_{\mathbb{R}^N} mu_k^2 \to \int_{\mathbb{R}^N} mu^2$ as $k \to \infty$. Moreover since $(u_k)$ converges weakly to $u$ in $V_q(\mathbb{R}^N)$, we have $\|u\|_q \leq \liminf \|u_k\|_q = \lambda_{1,q,m}$. Therefore $u$ realizes the infimum (2.2) and so $u$ is on the form $u = \alpha\phi_{1,q,m}$ with $\alpha \in \mathbb{R}$, $\alpha \neq 0$.

If now $m$ satisfies $(\mathbf{H_m'^2})$, note that $\int_{\mathbb{R}^N} m_1u_k^2 \to \int_{\mathbb{R}^N} m_1u^2$ as $k \to \infty$. Recall that $1 = \int_{\mathbb{R}^N} mu_k^2 = \int_{\mathbb{R}^N} m_1u_k^2 - \int_{\mathbb{R}^N} m_2u_k^2$. Thus

$$\int_{\mathbb{R}^N} m_2u^2 \leq \liminf \int_{\mathbb{R}^N} m_2u_k^2 = \int_{\mathbb{R}^N} m_1u^2 - 1.$$

Therefore $\int_{\mathbb{R}^N} mu^2 \geq 1$ and there exists $\beta \in ]0,1]$ such that $\int_{\mathbb{R}^N} m(\beta u)^2 = 1$. Moreover since $(u_k)$ converges weakly to $u$ in $V_q(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} [|\nabla u|^2 + qu^2] \leq \liminf \int_{\mathbb{R}^N} [|\nabla u_k|^2 + qu_k^2] = \lambda_{1,q,m}, \tag{2.3}$$

and from the variational characterization (2.2) of $\lambda_{1,q,m}$ we also have

$$\lambda_{1,q,m} \leq \int_{\mathbb{R}^N} [|\nabla(\beta u)|^2 + q(\beta u)^2] = \beta^2 \int_{\mathbb{R}^N} [|\nabla u|^2 + qu^2]. \tag{2.4}$$

From (2.3) and (2.4) we get $\beta^2 \geq 1$ and therefore $\beta = 1$. So here again $u$ realizes the infimum (2.2) and therefore $u$ is on the form $u = \alpha\phi_{1,q,m}$ with $\alpha \in \mathbb{R}$, $\alpha \neq 0$. □

## 3 The local antimaximum principle

In this section we consider the equation (1.1) where $m$ satisfies $(\mathbf{H'^1_m})$ or $(\mathbf{H'^2_m})$, $q$ satisfies $(\mathbf{H^1_q})$, $(\mathbf{H^2_q})$ and $f$ satisfies $(\mathbf{H^1_f})$. Let $u$ be a weak solution of (1.1). Recall that $u \in C^1(\mathbb{R}^N)$. First we recall the Picone identity.

**Lemma 3.1** *Let $\Omega$ be a domain in $\mathbb{R}^N$. For $\psi, u \in C^1(\Omega)$ with $\psi \geq 0$ and $u > 0$ in $\Omega$, we have $|\nabla\psi|^2 - \nabla u \cdot \nabla(\frac{\psi^2}{u}) \geq 0$ in $\Omega$.*

**Proposition 3.1** *If $f \geq 0$, $f \not\equiv 0$, then (1.1) has no solution if $\lambda = \lambda_{1,q,m}$ and has no nonnegative solution if $\lambda > \lambda_{1,q,m}$.*

**Proof:** First assume that $\lambda = \lambda_{1,q,m}$ and there exists a solution $u$ for (1.1). Multiplying (1.1) by $\phi_{1,q,m}$ as a test function, we obtain that $\int_{\mathbb{R}^N} f\phi_{1,q,m} = 0$ and so we get a contradiction since $f \geq 0$, $f \not\equiv 0$, $\phi_{1,q,m} > 0$.

Assume now that $\lambda > \lambda_{1,q,m}$ and there exists a nonnegative solution $u$ for (1.1). Let $R > 0$ and $c_R$ a positive constant sufficiently large such that $c_R + \lambda m - q \geq 0$ in $B_R$. Note that $-\Delta u + c_R u = (c_R + \lambda m - q)u + f \geq 0$ in $B_R$. Applying the strong maximum principle in $B_R$ (see [17, Theorem 8.19 p.198]) (or as in [15, 24] the Vázquez maximum principle given in [28, Theorems 1,5]) we obtain that $u > 0$ in $B_R$ for any $R$ sufficiently large and so $u > 0$ in $\mathbb{R}^N$.

Let now $(\psi_k)_k$ be a convergent sequence to $\phi_{1,q,m}$ in $V_q(\mathbb{R}^N)$, $\psi_k \geq 0$, $\psi_k \in D(\mathbb{R}^N)$. Applying the Picone identity, we get

$$\int_{\mathbb{R}^N} \left(|\nabla\psi_k|^2 - \nabla u \cdot \nabla(\frac{\psi_k^2}{u})\right) = \|\psi_k\|_q^2 - \lambda \int_{\mathbb{R}^N} m\psi_k^2 - \int_{\mathbb{R}^N} f\frac{\psi_k^2}{u} \geq 0.$$

Since $(\psi_k)_k$ is a convergent sequence to $\phi_{1,q,m}$ in $V_q(\mathbb{R}^N)$, we have

$$\|\psi_k\|_q^2 \to \|\phi_{1,q,m}\|_q^2 = \lambda_{1,q,m} \int_{\mathbb{R}^N} m\phi_{1,q,m}^2 \text{ as } k \to \infty.$$

If $m$ satisfies $(\mathbf{H_m'^1})$, note that $\psi_k \to \phi_{1,q,m}$ in $L^2(\mathbb{R}^N)$ as $k \to \infty$ and (at least for a subsequence still denoted by $(\psi_k)$) there exists $h \in L^2(\mathbb{R}^N)$ such that $\psi_k \to \phi_{1,q,m}$ a.e. in $\mathbb{R}^N$ and $|\psi_k| \leq h$ a.e. in $\mathbb{R}^N$ for all $k$. So, since $m \in L^\infty(\mathbb{R}^N)$, there exists a positive constant $C$ such that $|m\psi_k^2 - m\phi_{1,q,m}^2| \leq C(h^2 + \phi_{1,q,m}^2)$ a.e. in $\mathbb{R}^N$. Applying the Lebesgue dominated convergence Theorem, we deduce that

$$\int_{\mathbb{R}^N} m\psi_k^2 \to \int_{\mathbb{R}^N} m\phi_{1,q,m}^2 \text{ as } k \to \infty. \tag{3.1}$$

By the same way, if $m$ satisfies $(\mathbf{H_m'^2})$, recall that $N \geq 3$ and $H^1(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N)$ with a continuous embedding, so note that $\psi_k \to \phi_{1,q,m}$ in $L^{2^*}(\mathbb{R}^N)$ as $k \to \infty$ with $2^* = \frac{2N}{N-2}$ and (at least for a subsequence still denoted by $(\psi_k)$) there exists $h \in L^{2^*}(\mathbb{R}^N)$ such that $\psi_k \to \phi_{1,q,m}$ a.e. in $\mathbb{R}^N$ and $|\psi_k| \leq h$ a.e. in $\mathbb{R}^N$ for all $k$. So $|m\psi_k^2 - m\phi_{1,q,m}^2| \leq |m|(h^2 + \phi_{1,q,m}^2)$ a.e. in $\mathbb{R}^N$. Applying the Lebesgue dominated convergence Theorem, we still get (3.1).

Finally, note that $\frac{f\psi_k^2}{u} \geq 0$ and $\frac{f\psi_k^2}{u} \in L^1(\mathbb{R}^N)$ since $\psi_k$ has a compact support, $f \in L^\infty_{loc}(\mathbb{R}^N)$, $u \in L^\infty_{loc}(\mathbb{R}^N)$. By the Fatou lemma we get that

$$\int_{\mathbb{R}^N} \frac{f\phi_{1,q,m}^2}{u} \leq \liminf \int_{\mathbb{R}^N} \frac{f\psi_k^2}{u}.$$

So by the Lebesgue dominated convergence Theorem and Fatou Lemma, we obtain

$$(\lambda_{1,q,m} - \lambda) \int_{\mathbb{R}^N} m\phi_{1,q,m}^2 - \int_{\mathbb{R}^N} f\frac{\phi_{1,q,m}^2}{u} \geq 0.$$

And we get a contradiction since the first term of this estimate is negative and the second term is negative too. □

We give now the local antimaximum principle.

**Theorem 3.1** *Let $f \geq 0$, $f \not\equiv 0$. Then for any $R > 0$ there exists a positive constant $\delta = \delta(f, R) > 0$ such that for any $\lambda \in ]\lambda_{1,q,m}, \lambda_{1,q,m} + \delta[$, any solution $u$ of (1.1) is negative in $\overline{B}_R$.*

**Proof:** We follow [15, 24]. Assume by contradiction that for some $R > 0$ there exist $\lambda_k > \lambda_{1,q,m}$, $\lambda_k \searrow \lambda_{1,q,m}$, a solution $u_k$ of

$$(-\Delta + q)u_k = \lambda_k m u_k + f \text{ in } \mathbb{R}^N, \tag{3.2}$$

and $x_k \in \overline{B}_R$ such that $u_k(x_k) \geq 0$.

First we show that $\lim_{k\to\infty}\|u_k\|_q=\infty$. On the contrary, assume that $(\|u_k\|_q)_k$ is a bounded sequence. Therefore, from the compact embedding of $V_q(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$, for a subsequence, there exists $u\in V_q(\mathbb{R}^N)$ such that $(u_k)_k$ converges to $u$, weakly in $V_q(\mathbb{R}^N)$ and strongly in $L^2(\mathbb{R}^N)$. So for all $\phi\in D(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N}(\nabla u_k\cdot\nabla\phi+qu_k\phi)\to\int_{\mathbb{R}^N}(\nabla u\cdot\nabla\phi+qu\phi)\text{ as }k\to\infty$$

and by the Lebesgue dominated convergence Theorem

$$\int_{\mathbb{R}^N}mu_k\phi\to\int_{\mathbb{R}^N}mu\phi\text{ as }k\to\infty.$$

Therefore as $k\to\infty$, we get from (3.2) that $(-\Delta+q)u=\lambda_{1,q,m}mu+f$ in $\mathbb{R}^N$ which contradicts Proposition 3.1. So $\lim_{k\to\infty}\|u_k\|_q=\infty$.

Now set $v_k=\frac{u_k}{\|u_k\|_q}$. Then $v_k$ satisfies

$$(-\Delta+q)v_k=\lambda_k mv_k+\frac{f}{\|u_k\|_q}\text{ in }\mathbb{R}^N. \tag{3.3}$$

Since $(v_k)_k$ is a bounded sequence in $V_q(\mathbb{R}^N)$, as before, for a subsequence, there exists $v\in V_q(\mathbb{R}^N)$ such that $(v_k)_k$ converges to $v$, weakly in $V_q(\mathbb{R}^N)$ and strongly in $L^2(\mathbb{R}^N)$. And $v$ satisfies

$$(-\Delta+q)v=\lambda_{1,q,m}mv\text{ in }\mathbb{R}^N.$$

Since $\lambda_{1,q,m}$ is a simple eigenvalue then there exists $\beta\in\mathbb{R}$ such that

$$v=\beta\phi_{1,q,m}.$$

First note that if we multiply (3.3) by $\phi_{1,q,m}$ as a test function and if we integrate over $\mathbb{R}^N$, we get

$$\lambda_{1,q,m}\int_{\mathbb{R}^N}m\phi_{1,q,m}v_k=\lambda_k\int_{\mathbb{R}^N}m\phi_{1,q,m}v_k+\int_{\mathbb{R}^N}\frac{f}{\|u_k\|_q}\phi_{1,q,m}.$$

Therefore since $\int_{\mathbb{R}^N}\frac{f}{\|u_k\|_q}\phi_{1,q,m}>0$ and $\lambda_{1,q,m}<\lambda_k$ we have $\int_{\mathbb{R}^N}m\phi_{1,q,m}v_k<0$. So if the weight $m$ satisfies $(\mathbf{H_m'^1})$, passing to the limit we get that $\int_{\mathbb{R}^N}m\phi_{1,q,m}v\le 0$ and $\beta\le 0$. Therefore we will consider three cases for $\beta$ (and the case $\beta>0$ only when the weight $m$ satisfies $(\mathbf{H_m'^2})$).

If $\beta=0$ then $v=0$. Note that $\|v_k\|_q^2=\lambda_k\int_{\mathbb{R}^N}mv_k^2+\int_{\mathbb{R}^N}\frac{fv_k}{\|u_k\|_q}$ and $\|v_k\|_q^2\le\lambda_k\int_{\mathbb{R}^N}pv_k^2+\int_{\mathbb{R}^N}\frac{fv_k}{\|u_k\|_q}$ with $p:=m$ if $m$ satisfies $(\mathbf{H_m'^1})$ and $p:=m_1$ if $m$ satisfies $(\mathbf{H_m'^2})$. By the Lebesgue dominated convergence theorem, we get that $\int_{\mathbb{R}^N}pv_k^2\to 0$ and $\int_{\mathbb{R}^N}\frac{fv_k}{\|u_k\|_q}\to 0$ as $k\to\infty$. So we have $\|v_k\|_q\to 0$ as $k\to\infty$, which is impossible since $\|v_k\|_q=1$. Therefore $\beta\neq 0$.

If now $\beta < 0$ then $v < 0$ in $\mathbb{R}^N$. But $(v_k)_k$ converges to $v$ in $C^1_{loc}(\mathbb{R}^N)$ and uniformly on all ball $B_R$. So $v_k$ is negative in $\overline{B}_R$ for $k$ sufficiently large, which contradicts the existence of the sequence $x_k$.

So we consider the last case $\beta > 0$ (and in fact only when the weight $m$ satisfies $(\mathbf{H'^2_m})$). We will show that $v_k \geq 0$ i.e. $v_k^- \equiv 0$ in $\mathbb{R}^N$ for $k$ sufficiently large. On the contrary, assume that $v_k^- \not\equiv 0$. Multiplying (3.3) by $v_k^-$ and integrating over $\mathbb{R}^N$, we get that

$$0 < \|v_k^-\|_q^2 = \lambda_k \int_{\mathbb{R}^N} m(v_k^-)^2 - \int_{\mathbb{R}^N} \frac{f v_k^-}{\|u_k\|_q} \leq \lambda_k \int_{\mathbb{R}^N} m(v_k^-)^2.$$

So $r_k := \int_{\mathbb{R}^N} m(v_k^-)^2 > 0$. Moreover by the variational characterization of $\lambda_{1,q,m}$ we have

$$\frac{\|v_k^-\|_q^2}{\int_{\mathbb{R}^N} m(v_k^-)^2} \to \lambda_{1,q,m} \text{ as } k \to \infty \text{ i.e. } \lim_{k\to\infty} \|w_k\|_q^2 = \lambda_{1,q,m} \text{ with } w_k = \frac{1}{r_k^{1/2}} v_k^-.$$

So $(w_k)$ is a minimizing sequence for $\lambda_{1,q,m}$ in (2.2) and from the simplicity of the eigenvalue $\lambda_{1,q,m}$, using Proposition 2.1, for a subsequence, we deduce that $(w_k)_k$ converges to $\alpha\phi_{1,q,m}$ with $\alpha \in \mathbb{R}$, $\alpha \neq 0$, weakly in $V_q(\mathbb{R}^N)$ (and strongly in $L^2(\mathbb{R}^N)$). But $(v_k)_k$ converges to $\beta\phi_{1,q,m} > 0$ in $C^1_{loc}(\mathbb{R}^N)$. So $v_k$ is positive on the unit ball $B_1$ for $k$ sufficiently large and so $v_k^- \equiv 0$, $w_k \equiv 0$ on $B_1$. Thus $\alpha = 0$. Therefore we get a contradiction.

So $v_k \geq 0$ in $\mathbb{R}^N$ for $k$ sufficiently large and $v_k$ satisfies (3.3) with $\lambda_k > \lambda_{1,q,m}$. This contradicts Proposition 3.1 and this concludes the proof of the local antimaximum principle theorem. □

Note that we obtain the same kind of local antimaximum principle for $\lambda < \tilde{\lambda}_{1,q,m}$ since $\tilde{\lambda}_{1,q,m} = -\lambda_{1,q,-m}$ and the equation $(-\Delta + q)u = \lambda m u + f$ is equivalent to $(-\Delta + q)u = (-\lambda)(-m)u + f$. To conclude this section, as in [15], we give a result for the semi-global antimaximum principle.

**Proposition 3.2** *Assume that there exists $R_0 \in \mathbb{R}$, $R_0 > 0$ such that $m \leq 0$ in $B^C_{R_0} := \mathbb{R}^N \setminus B_{R_0}$. Assume also that $f \geq 0$, $f \not\equiv 0$, and there exists a constant $C \geq 0$ such that $f \leq -Cm\phi_{1,q,m}$ in $B^C_{R_0}$. Let $\delta = \delta(f, R_0)$ be given by Theorem 3.1. Then for any $\lambda \in ]\lambda_{1,q,m}, \lambda_{1,q,m} + \delta[$, any solution $u$ of (1.1) satisfies $u \leq \frac{C}{\lambda - \lambda_{1,q,m}} \phi_{1,q,m}$ in $\mathbb{R}^N$.*

**Proof:** Let $C' = \frac{C}{\lambda - \lambda_{1,q,m}}$ (with $\lambda \in ]\lambda_{1,q,m}, \lambda_{1,q,m} + \delta[$) and $v = u - C'\phi_{1,q,m}$. We want to prove that $v^+ = 0$ in $\mathbb{R}^N$. First note that $v^+ = 0$ in $B_{R_0}$ by Theorem 3.1. Moreover we have

$$(-\Delta + q)v = (\lambda - \lambda_{1,q,m}) m u + \lambda_{1,q,m} m v + f \text{ in } \mathbb{R}^N. \tag{3.4}$$

Multiplying (3.4) by $v^+$ and integrating over $\mathbb{R}^N$, since $v^+ = 0$ in $B_{R_0}$, we get

$$0 \leq \int_{B^C_{R_0}} [|\nabla v^+|^2 + q|v^+|^2] = \lambda_{1,q,m} \int_{B^C_{R_0}} m|v^+|^2 + \int_{B^C_{R_0}} [(\lambda - \lambda_{1,q,m}) m u + f] v^+.$$

Since $f \leq -Cm\phi_{1,q,m}$ in $B^C_{R_0}$ and $m \leq 0$ in $B^C_{R_0}$, we obtain

$$0 \leq \int_{B^C_{R_0}} [(\lambda - \lambda_{1,q,m})mu + f]v^+ \leq (\lambda - \lambda_{1,q,m}) \int_{B^C_{R_0}} m|v^+|^2 \leq 0.$$

Therefore $\int_{B^C_{R_0}} [|\nabla v^+|^2 + q|v^+|^2] = 0$ and $v^+ = 0$ in $B^C_{R_0}$. □

**Theorem 3.2** *Assume that there exists $R_0 \in \mathbb{R}$, $R_0 > 0$ such that $m \leq 0$ in $B^C_{R_0} := \mathbb{R}^N \setminus B_{R_0}$. Assume also that $f \geq 0$, $f \not\equiv 0$, $f$ with compact support. Then there exists $\delta := \delta(f)$ a positive constant such that for any $\lambda \in ]\lambda_{1,q,m}, \lambda_{1,q,m} + \delta[$, any solution $u$ of (1.1) is negative in $\mathbb{R}^N$.*

**Proof:** Let $R_1 \geq R_0$ such that *suppf* $\subset B_{R_1}$ and let $\delta := \delta(f, R_1)$ given by Theorem 3.1. Let $\lambda \in ]\lambda_{1,q,m}, \lambda_{1,q,m} + \delta[$ and $u$ a solution of (1.1). From Proposition 3.2 with $C = 0$ we get that $u \leq 0$ in $\mathbb{R}^N$. Moreover from Theorem 3.1 we have $u < 0$ in $\overline{B}_{R_1}$.

Let now $x \in \overline{B}^C_{R_1} := \mathbb{R}^N \setminus \overline{B}_{R_1}$ and $r > 0$ such that $B(x, r) \cap B_{R_1} \neq \emptyset$ and $B(x, r) \cap suppf = \emptyset$. Let $c_r$ be a positive constant such that $c_r + \lambda m - q > 0$ in $B(x, r)$ the open ball of center $x$ and radius $r$. Since $(-\Delta)(-u) + c_r(-u) = (c_r + \lambda m - q)(-u) \geq 0$ in $B(x, r)$, by the strong maximum principle, we get that $-u \equiv 0$ or $-u > 0$ in $B(x, r)$. Since $u < 0$ in $\overline{B}_{R_1}$ we deduce that $-u > 0$ in $B(x, r)$. So $u(x) < 0$ and this concludes the proof of the semi-global antimaximum principle. □

# 4 Study of a linear elliptic system

In this section, we study the antimaxium principle for the following system

$$(-\Delta + q_i)u_i = \lambda \left( m_i u_i + \sum_{j=1; j \neq i}^{n} m_{ij} u_j \right) + f_i \text{ in } \mathbb{R}^N, \; i = 1, \cdots, n, \tag{4.1}$$

where each of the potentials $q_i$ satisfy $(\mathbf{H^1_q})$-$(\mathbf{H^2_q})$, each of the weights $m_i$ satisfy the hypothesis $(\mathbf{H'^1_m})$ and each of the functions $f_i$ satisfy the hypothesis $(\mathbf{H^1_f})$. We denote by $M$ the $n \times n$-matrix given by $M = (m_{ij})$ with $m_{ii} = m_i$. We will consider the following hypotheses:

$(\mathbf{H^1_M})$ For all $i \neq j$, $m_{ij} \in L^\infty(\mathbb{R}^N)$ and $m_{ij} > 0$.

$(\mathbf{H^2_M})$ $M$ is a symmetric matrix.

$(\mathbf{H^3_M})$ $\Omega := \cap_{i=1}^n \Omega_i^+$ is an open subset of $\mathbb{R}^N$ with non zero measure and with $\Omega_i^+ := \{x \in \mathbb{R}^N, m_i(x) > 0\}$.

We also consider the following system:

$$(-\Delta + q_i)u_i = \lambda \left( m_i u_i + \sum_{j=1; j\neq i}^{n} m_{ij} u_j \right) \text{ in } \mathbb{R}^N, \ i = 1, \cdots, n. \tag{4.2}$$

We recall from [7] the existence of a positive and simple eigenvalue associated with a positive eigenfunction for (4.2).

**Theorem 4.1** *Assume that each of the potentials $q_i$ satisfy* $(\mathbf{H_q^1})$-$(\mathbf{H_q^2})$ *and each of the weights $m_i$ satisfy* $(\mathbf{H_m'^1})$. *Assume also that* $(\mathbf{H_M^1})$-$(\mathbf{H_M^3})$ *are satisfied. Then there exists a unique principal eigenvalue $\Lambda_{1,M} > 0$ associated with a positive eigenfunction $\Phi_{1,M} = (\phi_{1,M}, \cdots, \phi_{n,M}) \in V := V_{q_1}(\mathbb{R}^N) \times \cdots \times V_{q_n}(\mathbb{R}^N)$ for the system (4.2) (and $\phi_{i,M} > 0$ for all $i$). The eigenvalue $\Lambda_{1,M}$ is simple and verifies*

$$\Lambda_{1,M} = \inf \left\{ \frac{\sum_{i=1}^n \|u_i\|_{q_i}^2}{\sum_{i=1}^n \int_{\mathbb{R}^N} m_i u_i^2 + \sum_{i,j;i\neq j}^n \int_{\mathbb{R}^N} m_{ij} u_i u_j}, \ u = (u_1, \cdots, u_n) \in V \right.$$

$$\left. \text{such that } \sum_{i=1}^n \int_{\mathbb{R}^N} m_i u_i^2 + \sum_{i,j;i\neq j}^n \int_{\mathbb{R}^N} m_{ij} u_i u_j > 0 \right\}. \tag{4.3}$$

We recall from [7] the maximum principle for (4.1).

**Theorem 4.2** *Assume that each of the potentials $q_i$ satisfy* $(\mathbf{H_q^1})$-$(\mathbf{H_q^2})$ *and each of the weights $m_i$ satisfy* $(\mathbf{H_m'^1})$. *Assume also that* $(\mathbf{H_M^1})$-$(\mathbf{H_M^3})$ *are satisfied. Assume that $f_i \in L^2(\mathbb{R}^N)$ for all $i$. If $0 \le \lambda < \Lambda_{1,M}$, then the system (4.1) satisfies the maximum principle: if $f = (f_1, \cdots, f_n) \ge 0$, then $u_i \ge 0$ for all $i$ with $u = (u_1, \cdots, u_n)$ solution of (4.1).*

Now we study the local antimaximum principle for (4.1). As for one equation, note that any solution $u = (u_1, \cdots, u_n)$ of (4.1) satisfies $u \in (C^1(\mathbb{R}^N))^n$. We now extend Proposition 3.1 to the system (4.1).

**Proposition 4.1** *If $f_i \ge 0$, $f_i \not\equiv 0$ for all $i$, then (4.1) has no solution if $\lambda = \Lambda_{1,M}$ and has no nonnegative solution if $\lambda > \Lambda_{1,M}$.*

**Proof:** First assume that $\lambda = \Lambda_{1,M}$ and there exists a solution $u = (u_1, \cdots, u_n)$ for (4.1). Multiplying each equation of (4.1) by $\phi_{i,M}$ as a test function, integrating over $\mathbb{R}^N$ and adding all these equations, since $M$ is a symmetric matrix, we obtain that $\sum_{i=1}^n \int_{\mathbb{R}^N} f_i \phi_{i,M} = 0$ and so we get a contradiction since $f_i \ge 0$, $f_i \not\equiv 0$, $\phi_{i,M} > 0$.

Assume now that $\lambda > \Lambda_{1,M}$ and there exists a nonnegative solution $u = (u_1, \cdots, u_n)$ for (4.1) i.e. $u_i \ge 0$ for all $i$. Let $R > 0$ and $c_R$ a positive constant sufficiently large such that $c_R + \lambda m_i - q_i \ge 0$ in $B_R$ for any $i$. Note that for any $i$

$$-\Delta u_i + c_R u_i = (c_R + \lambda m_i - q_i)u_i + \lambda \sum_{j=1; j\neq i}^{n} m_{ij} u_j + f_i \ge 0 \text{ in } B_R.$$

Applying the strong maximum principle in $B_R$, since $\lambda > 0$, $m_{ij} > 0$, $u_j \geq 0$, $f_i \geq 0$, $f_i \not\equiv 0$, we obtain that $u_i > 0$ in $B_R$ for any $R$ sufficiently large and so $u_i > 0$ in $\mathbb{R}^N$.

Let now for each $i = 1, \cdots, n$ $(\psi_{ik})_k$ be a convergent sequence to $\phi_{i,M}$ in $V_{q_i}(\mathbb{R}^N)$, $\psi_{ik} \geq 0$, $\psi_{ik} \in D(\mathbb{R}^N)$. Applying the Picone identity, we get

$$\int_{\mathbb{R}^N} \left( |\nabla \psi_{ik}|^2 - \nabla u_i \cdot \nabla (\frac{\psi_{ik}^2}{u_i}) \right) =$$
$$\|\psi_{ik}\|_{q_i}^2 - \lambda \int_{\mathbb{R}^N} m_i \psi_{ik}^2 - \lambda \sum_{j=1; j\neq i}^{n} \int_{\mathbb{R}^N} m_{ij} u_j \frac{\psi_{ik}^2}{u_i} - \int_{\mathbb{R}^N} f_i \frac{\psi_{ik}^2}{u_i} \geq 0.$$

Therefore

$$\sum_{i=1}^{n} \int_{\mathbb{R}^N} \left( |\nabla \psi_{ik}|^2 - \nabla u_i \cdot \nabla (\frac{\psi_{ik}^2}{u_i}) \right) = \sum_{i=1}^{n} \|\psi_{ik}\|_{q_i}^2 - \lambda \sum_{i=1}^{n} \int_{\mathbb{R}^N} m_i \psi_{ik}^2$$
$$-\lambda \sum_{i,j=1; j\neq i}^{n} \int_{\mathbb{R}^N} m_{ij} u_j \frac{\psi_{ik}^2}{u_i} - \sum_{i=1}^{n} \int_{\mathbb{R}^N} f_i \frac{\psi_{ik}^2}{u_i} \geq 0.$$

Since $(\psi_{ik})_k$ is a convergent sequence to $\phi_{i,M}$ in $V_{q_i}(\mathbb{R}^N)$, we have

$$\|\psi_{ik}\|_{q_i}^2 \to \|\phi_{i,M}\|_{q_i}^2 = \Lambda_{1,M} (\int_{\mathbb{R}^N} m_i \phi_{i,M}^2 + \sum_{j=1; i\neq j}^{n} \int_{\mathbb{R}^N} m_{ij} \phi_{i,M} \phi_{j,M}) \text{ as } k \to \infty.$$

Passing to the limit by the Lebesgue dominated convergence Theorem and Fatou Lemma, we get

$$(\Lambda_{1,M} - \lambda) \sum_{i=1}^{n} \int_{\mathbb{R}^N} m_i \phi_{i,M}^2 - \sum_{i=1}^{n} \int_{\mathbb{R}^N} f_i \frac{\phi_{i,M}^2}{u_i}$$
$$+\Lambda_{1,M} \sum_{i,j=1; i\neq j}^{n} \int_{\mathbb{R}^N} m_{ij} \phi_{i,M} \phi_{j,M} - \lambda \sum_{i,j=1; j\neq i}^{n} \int_{\mathbb{R}^N} m_{ij} u_j \frac{\phi_{iM}^2}{u_i} \geq 0.$$

Since $\lambda > \Lambda_{1,M}$ and $\sum_{i,j=1; i\neq j}^{n} \int_{\mathbb{R}^N} m_{ij} \phi_{i,M} \phi_{j,M} > 0$ thus

$$(\Lambda_{1,M} - \lambda) \sum_{i=1}^{n} \int_{\mathbb{R}^N} m_i \phi_{i,M}^2 - \sum_{i=1}^{n} \int_{\mathbb{R}^N} f_i \frac{\phi_{i,M}^2}{u_i} - \lambda \sum_{i,j=1; i<j}^{n} \int_{\mathbb{R}^N} \frac{m_{ij}}{u_i u_j} (u_j \phi_{i,M} - u_i \phi_{j,M})^2 \geq 0.$$

And we get a contradiction since all the two first terms of this estimate are negative and the third term is nonpositive. $\square$

We give now the local antimaximum principle.

**Theorem 4.3** *Let $f = (f_1, \cdots, f_n)$, $f_i \geq 0$, $f_i \not\equiv 0$ for all $i$. Then for any $R > 0$ there exists a positive constant $\delta = \delta(f, R) > 0$ such that for any $\lambda \in ]\Lambda_{1,M}, \Lambda_{1,M} + \delta[$, any solution $u = (u_1, \cdots, u_n)$ of (4.1) is negative in $\overline{B}_R$ i.e. $u_i < 0$ in $\overline{B}_R$ for all $i$.*

**Proof:** Assume by contradiction that for some $R > 0$ there exist $\lambda_k > \Lambda_{1,M}$, $\lambda_k \searrow \Lambda_{1,M}$, a solution $u_k = (u_{1k}, \cdots, u_{nk})$ of

$$(-\Delta + q_i)u_{ik} = \lambda_k(m_i u_{ik} + \sum_{j=1;j\neq i}^{n} m_{ij}u_{jk}) + f_i \text{ in } \mathbb{R}^N,\ i = 1, \cdots, n \tag{4.4}$$

and $i_k \in \{1, \cdots, n\}$, $x_{i_k,k} \in \overline{B}_R$ such that $u_{i_k k}(x_{i_k,k}) \geq 0$.

First we show that $\lim_{k\to\infty} \|u_{ik}\|_{q_i} = \infty$ for at least one $i$. On the contrary, assume that $(\|u_{ik}\|_{q_i})_k$ is a bounded sequence for all $i$. Therefore, from the compact embedding of $V_{q_i}(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$, for a subsequence, there exists $u_i \in V_{q_i}(\mathbb{R}^N)$ such that $(u_{ik})_k$ converges to $u_i$, weakly in $V_{q_i}(\mathbb{R}^N)$ and strongly in $L^2(\mathbb{R}^N)$. Passing to the limit in (4.4) as in Theorem 3.1 we get

$$(-\Delta + q_i)u_i = \Lambda_{1,M}(m_i u_i + \sum_{j=1;j\neq i}^{n} m_{ij}u_j) + f_i \text{ in } \mathbb{R}^N,\ i = 1, \cdots, n$$

which contradicts Proposition 4.1. So $\lim_{k\to\infty} \|u_{ik}\|_{q_i} = \infty$ for at least one $i$.

Now set $v_{jk} = \frac{u_{jk}}{\sum_{i=1}^n \|u_{ik}\|_{q_i}}$ for all $j$. Then $v_{ik}$ satisfies

$$(-\Delta + q_i)v_{ik} = \lambda_k(m_i v_{ik} + \sum_{j=1;j\neq i}^{n} m_{ij}v_{jk}) + \frac{f_i}{\sum_{i=1}^n \|u_{ik}\|_{q_i}} \text{ in } \mathbb{R}^N,\ i = 1, \cdots, n. \tag{4.5}$$

Since $(v_{ik})_k$ is a bounded sequence in $V_{q_i}(\mathbb{R}^N)$, as before, for a subsequence, there exists $v_i \in V_{q_i}(\mathbb{R}^N)$ such that $(v_{ik})_k$ converges to $v_i$, weakly in $V_{q_i}(\mathbb{R}^N)$ and strongly in $L^2(\mathbb{R}^N)$. And $v = (v_1, \cdots, v_n)$ satisfies

$$(-\Delta + q_i)v_i = \Lambda_{1,M}(m_i v_i + \sum_{j=1;j\neq i}^{n} m_{ij}v_j) \text{ in } \mathbb{R}^N,\ i = 1, \cdots, n.$$

Since $\Lambda_{1,M}$ is a simple eigenvalue then there exists $\beta \in \mathbb{R}$ such that

$$v = \beta\Phi_{1,M} \text{ i.e. for all } i = 1, \cdots, n,\ v_i = \beta\phi_{i,M}.$$

We will consider three cases for $\beta$.

If $\beta = 0$ then $v = 0$. Note that for any $i = 1, \cdots, n$,

$$\|v_{ik}\|_{q_i}^2 = \lambda_k \int_{\mathbb{R}^N} m_i v_{ik}^2 + \lambda_k \sum_{j=1;j\neq i}^{n} \int_{\mathbb{R}^N} m_{ij}v_{jk}v_{ik} + \int_{\mathbb{R}^N} \frac{f_i v_{ik}}{\sum_{i=1}^n \|u_{ik}\|_{q_i}}.$$

By the Lebesgue dominated convergence Theorem, we get that $\|v_{ik}\|_{q_i} \to 0$ as $k \to \infty$ for all $i$, which is impossible since $\sum_{i=1}^n \|v_{ik}\|_{q_i} = 1$. Therefore $\beta \neq 0$.

If now $\beta < 0$ then $v < 0$ in $\mathbb{R}^N$. But for any $i$, $(v_{ik})_k$ converges to $v_i$ in $C^1_{loc}(\mathbb{R}^N)$. So $v_{ik}$ is negative in $\overline{B}_R$ for $k$ sufficiently large, which contradicts the existence of the sequence $x_{i_k,k}$.

So we consider the last case $\beta > 0$. We will show that for any $i$, $v_{ik} \geq 0$ i.e. $v_{ik}^- \equiv 0$ in $\mathbb{R}^N$ for $k$ sufficiently large. On the contrary, assume that there exists $i_0$ such that $v_{i_0k}^- \not\equiv 0$. Denote by

$$D(u) := \sum_{i=1}^n \int_{\mathbb{R}^N} m_i u_i^2 + \sum_{i,j;i\neq j}^n \int_{\mathbb{R}^N} m_{ij} u_i u_j \text{ for } u = (u_1, \cdots, u_n) \in V.$$

Multiplying (4.5) by $v_{ik}^-$ and integrating over $\mathbb{R}^N$, we get that

$$0 < \sum_{i=1}^n \|v_{ik}^-\|_{q_i}^2 = \lambda_k \Big(\sum_{i=1}^n \int_{\mathbb{R}^N} m_i (v_{ik}^-)^2 + \sum_{i,j=1;i\neq j}^n \int_{\mathbb{R}^N} m_{ij} v_{jk}^- v_{ik}^-\Big)$$

$$-\lambda_k \sum_{i,j=1;i\neq j}^n \int_{\mathbb{R}^N} m_{ij} v_{jk}^+ v_{ik}^- - \sum_{i=1}^n \int_{\mathbb{R}^N} \frac{f_i v_{ik}^-}{\sum_{i=1}^n \|u_{ik}\|_{q_i}}.$$

So $0 < \sum_{i=1}^n \|v_{ik}^-\|_{q_i}^2 \leq \lambda_k D(v_k^-)$ with $v_k^- = (v_{1k}^-, \cdots, v_{nk}^-)$ and therefore $D(v_k^-) > 0$ and $\Lambda_{1,M} = \lim_{k\to\infty} \frac{\|v_k^-\|_V^2}{D(v_k^-)}$ with $\|v_k^-\|_V^2 = \sum_{i=1}^n \|v_{ik}^-\|_{q_i}^2$.

Let $w_k = \frac{1}{D(v_k^-)^{1/2}} v_k^-$. So $(w_k)$ is a minimizing sequence for (4.3) and from the simplicity of the eigenvalue $\Lambda_{1,M}$, for a subsequence, we deduce that $(w_k)_k$ converges to $\alpha\Phi_{1,M}$ with $\alpha \in \mathbb{R}$, $\alpha \neq 0$, weakly in $V(\mathbb{R}^N)$ (and strongly in $(L^2(\mathbb{R}^N))^n$). Indeed first note that $(w_k)$ is a bounded sequence in $V$ so there exists $w \in V$ such that $(w_k)$ converges to $w$ weakly in $V$ and strongly in $(L^2(\mathbb{R}^N))^n$. Note also that $D(w_k) = 1$ and so $\frac{\|w_k\|_V^2}{D(w_k)} \to \Lambda_{1,M}$ as $k \to \infty$. Moreover $D(w_k) \to D(w)$ as $k \to \infty$ and $\|w\|_V \leq \liminf \|w_k\|_V = \sqrt{\Lambda_{1,M}}$ since $(w_k)$ converges weakly to $w$ in $V$. So using the variational characterization (4.3) of $\Lambda_{1,M}$ we get that $\frac{\|w\|_V^2}{D(w)} = \Lambda_{1,M}$. Thus $w$ realizes the infimum of $\Lambda_{1,M}$ and from the simplicity of the eigenvalue $\Lambda_{1,M}$ we deduce the existence of a real $\alpha \neq 0$ such that $w = \alpha\Phi_{1,M}$.

But $(v_k)_k$ converges to $\beta\Phi_{1,M} > 0$ in $(C^1_{loc}(\mathbb{R}^N))^n$. So for all $i$, $v_{ik}$ is positive on the unit ball $B_1$ for $k$ sufficiently large and so $v_{ik}^- \equiv 0$, $w_{ik} \equiv 0$ on $B_1$. Thus $\alpha = 0$. Therefore we get a contradiction.

So $v_{ik} \geq 0$ in $\mathbb{R}^N$ for all $i = 1, \cdots, n$ and for $k$ sufficiently large and $v_{ik}$ satisfies (4.5) with $\lambda_k > \Lambda_{1,M}$. This contradicts Proposition 4.1 and this concludes the proof of the local antimaximum principle theorem. □

These results can be extended to the system (4.1) with weights $m_i$ satisfying $(\mathbf{H'^2_m})$ (see [9] for the existence of a principal, positive and simple eigenvalue $\Lambda_{1,M}$).

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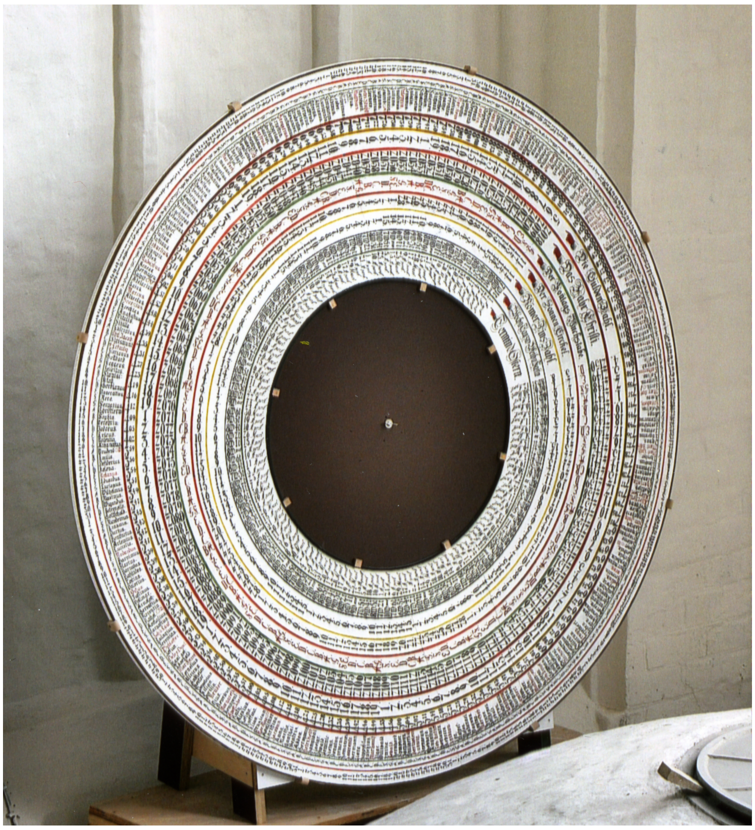
Klaus – Dieter Drews

# Betrachtungen zur Osterfestterminierung – auch als Reverenz für die Astronomische Uhr in Rostocks St.-Marien-Kirche und ihre neue Kalenderscheibe

Weil die jetzige Kalenderscheibe der St.-Marien-Uhr in denjenigen Angaben, welche sich auf zugeordnete Jahre beziehen, die Zeitspanne von 1885 nur bis 2017 erfaßt, ist für die wiederum 133 Jahre von 2018 bis 2150 bereits eine neue solche Scheibe gefertigt worden, als Deckplatte für die alte, und steht vorläufig separat in der Kirche. Die Vorlage der neuen Beschriftung wurde von Herrn M. Schukowski erstellt, unter Berücksichtigung einer Modifikation in den Rostocker Sonnenaufgangszeiten (s. [4]). Alle Angaben sowohl der neuen als auch der alten Kalenderscheibe enthält umfassend dokumentiert das jüngst erschienene Heft [11]. Wir werden die Scheibe darum hier nicht in Gänze beschreiben, sondern im wesentlichen die für unsere Zwecke prägnanten Teile als Anregung heranziehen.

*Neue Kalenderscheibe in der Werkstatt, Foto M. Mannewitz*

Im innersten Kreis der neuen Scheibe stehen die Daten der Ostersonntage von 2018 bis 2150. Die Angabe dieser Festtagsdaten sei uns Anlaß, Grundlagen ihrer Festlegungen zu erläutern, als notwendige Rekapitulation, um Gründe und sich ergebende Folgerungen darstellen zu



*Auf einer Empore in St.-Marien, Foto M. Mannewitz*

können. Wir werden keine fertige ‚Osterformel' heranziehen, unsere Ausführungen halten jedoch alles bereit, um jederzeit einen beliebigen Ostertermin (mit der Hand) bestimmen zu können, unser Blick hier ist aber eher auf den Ablauf der Termine in gewissen Zeitabschnitten und die ursächlichen Zusammenhänge gerichtet. – Den Zugang zum aufgegriffenen Thema hat mir schon vor Jahren die Arbeit [2] von S. Deschauer wesentlich erleichtert.

## 1 Tages- und Sonntagsbuchstaben

Die Tage des Jahres vom 1. Januar bis zum 31. Dezember, unter Aussparung des 29. Februars, bilden den äußersten Kreisring auf der Scheibe. Sie sind im zweiten Kreis mit den sieben Tagesbuchstaben A bis G zyklisch wiederkehrend gekennzeichnet, vom 1. Januar mit A, dem 2. mit B usw. bis zum 31. Dezember auch wieder mit A. Weiter innen findet man zu jedem Jahr (der erfaßten Zeitspanne) den Sonntagsbuchstaben, alle Tage dieses Buchstabens sind im besagten Jahr Sonntage. In Schaltjahren müssen es zwei Sonntagsbuchstaben sein, der erste Buchstabe gültig bis zum 28. Februar, der zweite, zyklisch alphabetisch vorhergehende, gültig ab 1. März; durch den zu berücksichtigenden 29. Februar liegen die Daten der Sonntage danach nämlich einen Tag eher. Die Bestimmung von Sonntagsterminen ist eines der Erfordernisse zur Osterfestterminierung, wozu die Sonntagsbuchstaben dienen können.

Unmittelbar vor der Scheibe stehend oder auch unter Vorlage des genannten Heftes [11] läßt sich für jedes Datum der erfaßten Zeitspanne der zugehörige Wochentag bestimmen:

Den Sonntagsbuchstaben des gewünschten Jahres aufsuchen und dann den Wochentag des gewünschten Datums vom nächstgelegenen Sonntag aus abzählen.

Vor der Uhr stehend wird diese Möglichkeit ab 2018 für zurückliegende Daten (evtl. den eigenen Geburtstag) wegen der fehlenden Sonntagsbuchstaben vielleicht zunächst vermißt werden. Aber:

> Auch für Jahre außerhalb der Zeitspanne auf der Scheibe lassen sich (mit der Scheibe!) zugehörige Sonntagsbuchstaben bestimmen.

Nach jedem 31. Dezember und nach jedem 29. Februar wechselt nämlich der Sonntagsbuchstabe in der Folge A bis G zyklisch um eine Position zurück. In 28 Jahren mit 7 Schalttagen ergeben sich 35 Wechsel, ein Vielfaches von 7 Wochentagen:

> Nach 28 Jahren wiederholt sich die Folge der Sonntagsbuchstaben.

Auf welchen Wochentag fiel der 27. April 1942? Es ist das Datum der letzten von vier aufeinanderfolgenden Nächten, in denen Rostock bombardiert wurde, 2012 gab es nach genau 70 Jahren ein Gedenkkonzert in der St.-Marien-Kirche, die damals – mit ihrer Uhr – vor der Zerstörung bewahrt blieb:

$1942 + 3 \cdot 28 = 2026$, Sonntagsbuchstabe D, der 27. April hat E, es war ein Montag (drei Wochen nach Ostermontag 1942, was vorläufig natürlich noch die jetzige Scheibe zeigt). Wenn später einmal jemand den Wochentag des 21. März 2185 wissen möchte (es wäre auch der 500. Geburtstag Johann Sebastian Bachs), so ist $2185 - 2 \cdot 28 = 2129$, So-Buchstabe B, der 21. März hat C, es wird ein Montag sein.

> Zu diesem Werkzeug muß aber die nachfolgende Ergänzung beachtet werden.

Liegt in 28 Jahren ein Säkularjahr ohne Schalttag (z. B. 1800, 1900, 2100), so entfällt eine der oben erwähnten Zurücksetzungen. Der 28 Jahre später abgelesene So-Buchstabe ist also eine Position zu wenig zurückgesetzt, für das Ausgangsjahr gilt der um eine weitere Position zurückgesetzte.

Weiter hinten werden wir uns z. B. für die Sonntage Ende März in den Jahren 1731, 1739, 1742 bzw. 1723 interessieren. Addiert man jeweils $11 \cdot 28 (= 308)$, so ergeben sich 2039, 2047, 2050 und 2031 mit den So-Buchstaben B, F, B und E; diese müssen nun je um zwei Positionen (1800, 1900) zurückgesetzt werden auf G, D, G, C, und entsprechen dann dem 25., 29., 25. bzw. 28. März.

## 2 Die Wurzeln der Osterfestterminierung

Allenthalben lexikalisch überliefert ist die Bestimmung des Konzils von Nizäa im Jahre 325, Ostern falle immer auf den ersten Sonntag nach dem ersten Frühlingsvollmond. Ostern, das Fest des Gedenkens an die Auferstehung Christi, hat für die Christenheit eine zentrale Bedeutung. Sein bewegliches Datum bestimmt auch die Daten für andere Feste des Jahres, wie Christi Himmelfahrt, 40 Tage ab Ostersonntag, Pfingsten, 50 Tage ab oder 7 Wochen nach Ostersonntag, und in evangelischer Zählung den Sonntag Trinitatis, als den Sonntag nach Pfingsten, und sodann die Anzahl der Sonntage nach Trinitatis, vom 1. bis maximal zum 27. Sonntag nach Trinitatis, jedenfalls bis zum letzten Sonntag des Kirchenjahres, dem Sonntag vor dem 1. Advent. Die zitierte Festlegung des Ostertermins gibt allerdings keinen Eindruck vom langen Bemühen um eine Einigung darauf und um eine schließlich allgemeine Akzeptanz, worüber in [1] ausführlich berichtet wird.

Im Jahre 525 hat Dionysius Exiguus die Grundlagen für eine formale Bestimmung des Osterdatums gelegt; sie basieren einerseits auf Erkenntnissen schon des Altertums und sind andererseits (unter einigen dann nötigen Ergänzungen) weiterhin gültig. Bereits 400 v. Chr. kannte Meton den 19jährigen Mondzyklus: Nach 19 Jahren (von je $365,25d$) steht der 235. Vollmond wieder an der Ausgangsposition! Wegen $19 \cdot 365,25d/235 = 29,5308\ldots d$ hat man in $29,53d$ einen exzellenten Näherungswert für eine *Lunation*, der Zeit zwischen zwei identischen Mondphasen, auch *synodischer* Monat genannt (heute gilt $29,530\,588\,853d$).

> Das Datum des ersten Frühlingsvollmonds heißt *Ostergrenze*,
> Frühlingsanfang wird auf den 21. März festgelegt; die Ostergrenzen liegen daher im Intervall der 30 Tage vom 21. März bis zum 19. April.

Nun gilt $$-11d < -10,89d = 12 \cdot 29,53d - 365,25d$$
und $$13 \cdot 29,53d - 325,25d = 18,64d < 19d\,.$$

> Von einem Jahr zum nächsten kommt die Ostergrenze also entweder nach 12 Lunationen und liegt dann (notwendig auf ganze Tage gerundet) 11 Kalendertage eher oder nach 13 Lunationen und dann 19 Tage später.

Für den 19jährigen Mondzyklus war ein entsprechender Ostergrenzen-Zyklus zu finden. Dabei ist schon klar:

> Liegt eine Ostergrenze im April, so liegt die des Folgejahres 11 Tage eher, liegt sie dagegen im März, so liegt die des Folgejahres 19 Tage später.

Denn im April wäre ‚$19d$ später' nach dem 19. April, im März ‚$11d$ eher' vor dem 21. März.

Man setzt
$$a = \text{mod (Jahreszahl, 19)}.$$

($a$ ist der kleinste nicht negative Rest bei Division der Jahreszahl durch 19.) In $a+1$ hat man die *Goldene Zahl* des Jahres, als „Die Gülden Zahl" findet sie sich auf der Kalenderscheibe in einem Ring direkt neben den Jahreszahlen.

Dionysius wählte das Jahr 532, für das $a = 0$ gilt, als Anfang des Zyklus und dazu die Ostergrenze 5. April als damalige astronomische Gegebenheit.

> Die Festlegung der Ostergrenze des Jahres 532 auf den 5. April ist Wurzel für sämtliche Osterfestterminierungen.

Für ‚Ostergrenze' möge hinfort die profane Abkürzung OG erlaubt sein.

Zunächst erhält man mit den soeben genannten Regeln den OG-Zyklus ab 532:

| $a$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| OG | 5 | 25 | 13 | 2 | 22 | 10 | 30 | 18 | 7 | 27 | 15 | 4 | 24 | 12 | 1 | 21 | 9 | 29 | 17 |
| | A | M | A | A | M | A | M | A | A | M | A | A | M | A | A | M | A | M | A. |

Dieser OG-Zyklus gilt im Julianischen Kalender; aber aus ihm folgen später alle weiteren OG-Zyklen im Gregorianischen Kalender.

Es sind hier 7 März- und 12 Apriltermine, mit jeweils 13 bzw. 12 Lunationen bis zum Folgetermin, zusammen die 235 Lunationen in 19 Jahren. Beim Schließen des Zyklus von $a = 18$ auf $a = 0$ kommt die OG übrigens $12d$ eher, statt der sonst üblichen $11d$. Dies Resultat (auch *Mondsprung* genannt) wird hingenommen, weil alle Termine auf volle Tage gerundet sind, die Festsetzungen grundsätzlich formal geschehen und sich nicht mehr direkt an astronomischen Gegebenheiten orientieren.

Folgende Besonderheiten müssen festgehalten werden:

> Für $a = 11, 12, \ldots, 18$ ist $\text{OG}(a) = \text{OG}(a-11) - 1d$.
> Für die verbleibenden $a = 8, 9, 10$ differiert $\text{OG}(a)$ von den ihr nächstliegenden, das sind $\text{OG}(a-8)$ und $\text{OG}(a+8)$, um $2d$ bzw. $-2d$.
> Es gibt im Zyklus keine drei aufeinanderfolgenden Daten.

(Wegen $-11 \equiv 19 \mod 30$ folgen die Beobachtungen auch aus $11 \cdot 19 \equiv -1 \mod 30$ bzw. $8 \cdot 19 \equiv 2 \mod 30$.)

## 3 Zum Mondlauf, Mondschaltungen

Eine besondere Würdigung verdient die Wiedergabe des Mondlaufes an der Astronomischen Uhr. Äußerlich sichtbar wandert ein Mondsymbol, befestigt an einer kreisförmigen Scheibe, im Gegenuhrzeigersinn über 12 Tierkreiszeichen. Darüber dreht sich gleichsinnig eine konzentrische Scheibe mit einem entsprechenden Sonnensymbol, aber zusätzlich mit einem kreisförmigen Ausschnitt, durch den Mondphasen sichtbar werden, herrührend von einer Bemalung der darunterliegenden ‚Mondscheibe'.

*Vollmond, 30.7.2007 5:00*
*Mondsymbol im Wassermann*

*Sonnensymbol*
*im Löwen*

*Letztes Viertel, 6.8.2007 15:00*
*Mondsymbol im Stier*

*Fotos M. Berger*

Das Uhrwerk läßt das Sonnensymbol genau einmal in 365 Tagen umlaufen. Die Frage des Schalttages erörtern wir hier nicht, es sei, auch für weitere im folgenden erwähnte Details, auf den Artikel [3] dieser Schriftenreihe hingewiesen.

Für den Lauf des Mondsymbols in bezug auf das Sonnensymbol müßte das Uhrwerk den synodischen Monat realisieren, d.h. $29,530\,588\,853d$ oder $29d12h44m$ (um etwa $2,87s$ gerundet). Obwohl nun im Rahmen der gewählten Konstruktion des Uhrwerks für das Antriebsverhältnis des Mondes ein beidseitig bester Näherungswert (im Sinne der Kettenbruchtheorie) vorliegt, beträgt der synodische Monat des Uhrwerks $29d13h6m$ (um ca. $6s$ gerundet), er ist etwa $22m$ zu groß. Sichtbar wird dieses permanente Verlieren nicht leicht, und wenn überhaupt, dann nur mit genauer Kenntnis der wahren Stellung des Naturmondes. Um die 4jährigen durchschnittlichen Abweichungen so gering wie möglich zu halten, hat sich, beginnend ab 2004, eine 4jährliche Justierung des Uhrmondes bewährt.

Nun aber offenbart der mechanisch regelmäßige Lauf des Uhrmondes, wenn man z. B. die genauen Zeiten des Vollmondes in der Natur und auf der Uhr vergleicht, daß der theoretische synodische Monat ein Durchschnittswert ist, in Wahrheit schwanken die Lunationen des Naturmondes beträchtlich. (Eine qualitative Erklärung hierfür liefern die Keplerschen Gesetze, s. auch dazu [3].) In nachfolgender kleiner Aufstellung zeigt sich für 2007 eine Folge von verlängerten Lunationen, wodurch der Rückstand des Uhrmondes (sogar) abnimmt, für 2009 gibt es eher moderate Abweichungen und Rückstandsänderungen, für 2012 eine Folge von verkürzten Lunationen, wodurch der Rückstand des Uhrmondes drastisch zunimmt.

| Wahrer Vollmond | | Lunation (theoretisch 29d 12h 44m) | Vollmond der Uhr | Rückstand |
|---|---|---|---|---|
| 3.1.2007 | 14:56 MEZ | 29d 13h 33m | 4.1. 9:14 | 18h 18m |
| 2.2. | 6:44 | 15h 48m | 2.2. 22:20 | 15h 36m |
| 4.3. | 0:15 | 17h 31m | 4.3. 11:26 | 11h 11m |
| 2.4. | 19:13 MESZ | 17h 58m | 3.4. 0:32 | 5h 19m |
| 2.5. | 12:08 | 16h 55m | 2.5. 13:38 | 1h 30m |
| 1.6. | 3:02 | 14h 54m | 1.6. 2:44 | – 18m |
| 30.6. | 15:47 | 12h 45m | 30.6. 15:50 | + 3m |
| 30.7. | 2:46 | 10h 59m | 30.7. 4:57 | 2h 11m |
| 11.3.2009 | 3:38 MEZ | 29d 11h 48m | 11.3. 7:44 | 4h 6m |
| 9.4. | 16:57 MESZ | 12h 19m | 9.4. 20:50 | 3h 53m |
| 9.5. | 6:02 | 13h 5m | 9.5. 9:56 | 3h 54m |
| 7.6. | 20:13 | 14h 11m | 7.6. 23:02 | 2h 49m |
| 7.7. | 11:22 | 15h 9m | 7.7. 12:08 | 46m |
| 6.8. | 2:56 | 15h 34m | 6.8. 1:14 | – 1h 42m |
| 4.9. | 18:03 | 15h 7m | 4.9. 14:20 | – 3h 43m |
| 8.3.2012 | 10:38 MEZ | 29d 11h 46m | 8.3. 12:29 | 1h 51m |
| 6.4. | 21:17 MESZ | 9h 39m | 7.4. 1:35 | 4h 18m |
| 6.5. | 5:34 | 8h 17m | 6.5. 14:41 | 9h 7m |
| 4.6. | 13:10 | 7h 36m | 5.6. 3:48 | 14h 38m |
| 3.7. | 20:50 | 7h 40m | 4.7. 16:54 | 20h 4m |
| 2.8. | 5:26 | 8h 36m | 2.8. 10:45 Neujustier. | 5h 19m |
| 31.8. | 15:57 | 10h 31m | 31.8. 23:52 | 7h 55m |

Die Ungleichmäßigkeit im Lauf des Natur-Vollmondes hat keinen Einfluß auf den Ostertermin, denn dieser bestimmt sich formal aus den Ostergrenzen. (Die heutigen OG, s. Abschnitt 4, sind für 2009 ($a = 14$) der 10.4., vgl. oben den 9.4., für 2012 ($a = 17$) der 7.4., vgl. oben den 6.4.; hier hätte dies allerdings sowieso keinen Einfluß auf die Termine für die nachfolgenden Ostersonntage am 12.4. 2009 bzw. 8.4. 2012.)

Aus dem generellen Lauf des Naturmondes ergibt sich eine erste Modifikation der starren Dionysischen Ostergrenzen. Betrachtet man den Meton-Zyklus von 235 Lunationen in je

19 Jahren unter moderner Genauigkeit, so erhält man

$$235 \cdot 29{,}530588\ldots d - 19 \cdot 365{,}25d = -0{,}061820\ldots d\,.$$

Um (absolut) diese Zeitspanne tritt im Vergleich mit $19 \cdot 365{,}25$ Tageslängen der 235. Vollmond eher ein. Dieser Tagesanteil bedingt für den Mondlauf in bezug auf den gleichförmigen Ablauf der Kalendertage eine

Rückdatierung der Ostergrenzen, genannt *Mondschaltungen.*
Die Gesamtzahl der Mondschaltungen sei mit $T$ bezeichnet.
Die Gregorianische Kalenderreform im Jahre 1582 legte dazu fest:

Ab 1800 erfolgen in je 2500 Jahren Rückdatierungen von $8d$, und zwar stets in Säkularjahren, 7 mal nach je 300 Jahren (2100 bis 3900) und dann einmal nach 400 Jahren (4300); eine Begründung zeigt $2500 \cdot (-0{,}06182\ldots d)/19 = -8{,}1\ldots d$.

Schon vorher erfolgte 1582 selbst eine Rückdatierung der Ostergrenzen um $3d$ und für 1800 wurde $1d$ vorgesehen; denn es ist $(1582-532)\cdot(-0{,}06182\ldots d)/19 = -3{,}4\ldots d$ und $(1800-532)\cdot(-0{,}06182\ldots d)/19 = -4{,}1\ldots d$. Für 1800 gilt also $T = 4$.

(Geringfügig voreilig erscheinen diese Korrekturen in $2500\cdot 365{,}2425d$ (Gregorianische Jahre) statt in $2500 \cdot 365{,}25d$.)

## 4 Sonnenschaltungen, Ostergrenzen im Gregorianischen Kalender

Im Laufe der Jahrhunderte seit Dionysius hatte sich der Termin 21. März in Richtung Sommer verschoben, weil ja das Julianische Jahr von $365{,}25d$ zu lang ist gegenüber dem Tropischen Jahr von $365{,}2422d$ (leicht gerundet), dies die Dauer zwischen zwei aufeinanderfolgenden Passagen der Sonne durch den Frühlingspunkt, dem Frühlingsanfang.

In der Gregorianischen Reform von 1582 entfielen erstens im Kalender $10d$, auf Do, 4.10. 1582 folgte Fr., 15.10. 1582, der Frühlingsanfang sollte dadurch zukünftig wieder auf den 21. März fallen.

Zweitens entstand das Gregorianische Jahr von $365{,}2425d$ Länge, weil in 400 Jahren jetzt stets 3 Schalttage entfallen ($-3/400 = -0{,}0075$), immer in Säkularjahren, deren Jahreszahl nicht durch 400 teilbar ist. (Auffällig ist: $(1582 - 532) \cdot (-0{,}0075) = -7{,}875$, eher 8 als 10 Tage hätten so ausfallen müssen, demnach war 532 astronomisch schon vor dem 21. März Frühlingsanfang gewesen.)

Aber drittens gibt es Auswirkungen auf die Daten der Ostergrenzen:

> Je ausfallendem Schalttag erhöhen sich die Kalenderdaten der OG um $1d$.
> Die Gesamtzahl dieser sogenannten *Sonnenschaltungen* heiße $S$, 1582 ist $S = 10$.
> Um $(S-T)d$ sind gegenüber den OG von 532 die Daten der neuen OG zu erhöhen, dabei zyklisch immer so, daß sie in die Zeitspanne 21. März bis 19. April fallen.

Für unsere Zwecke notieren wir die OG-Zyklen ab dem Jahr 1800, dem Anfangsjahr des Zyklus der Mondschaltungen, sowie auch ab 1900. Für 1800 gilt $S = 12$ (1700, 1800 ohne Schalttag), $T = 4$, mithin $S - T = 8$. Dies galt aber auch schon ab 1700, weil sich 1800 sowohl $S$ als auch $T$ erhöhen. 1900 dagegen erhöht sich $S - T$ um 1 auf 9, denn es wird $S = 13$, aber 2000 ändern sich weder $S$ noch $T$, 2100 werden $S = 14$ und $T = 5$, aber $S - T$ bleibt 9, darum gelten die OG von 1900 bis 2199 unverändert, nicht mehr 2200.

Dies sind die angekündigten OG-Zyklen, der von 532 erhöht um $8d$, dann nochmals um $1d$:

| $a$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1700, 1800 | 13 | 2 | 22 | 10 | 30 | 18 | 7 | 27 | 15 | 4 | 24 | 12 | 1 | 21 | 9 | 29 | 17 | 6 | 26 |
| $S = 11, 12$ | A | A | M | A | M | A | A | M | A | A | M | A | A | M | A | M | A | A | M |
| $T = \ 3,\ 4$ | | | | | | | | | | | | | | | | | | | |
| 1900, 2000, | 14 | 3 | 23 | 11 | 31 | $19^-$ | 8 | 28 | 16 | 5 | 25 | 13 | 2 | 22 | 10 | 30 | $18^-$ | 7 | 27 |
| 2100 | A | A | M | A | M | A | A | M | A | A | M | A | A | M | A | M | A | A | M. |
| $S = 13, 13, 14$ | | | | | | | | | | | | | | | | | | | |
| $T = \ 4,\ 4,\ 5$ | | | | | | | | | | | | | | | | | | | |

Man erhält die Folgen auch weiterhin sukzessiv aus der OG(0), auf einen Apriltermin folgt ein $11d$ früherer, auf einen Märztermin ein $19d$ späterer (vgl. Abschnitt 2).

> Es gibt im Gregorianischen Kalender 30 verschiedene OG-Zyklen, charakterisiert je schon durch die OG(0), die auf die 30 Tage vom 21. März bis zum 19. April fallen kann.

Unversehens stellten sich Probleme ein, die dann im Sinne der Tradition gelöst wurden. Die OG kann nämlich nun, anders als bei Dionysius, auch noch auf den 19. April fallen, und ist dieser ein Sonntag, so wäre Ostern erst am 26. April. Das aber bleibe ausgeschlossen und spätester Ostertermin der 25. April, darum erfolgten *Sonderbestimmungen*:

> Die OG 19. April wird zur Bestimmung des Ostertermins in den 18. April gewandelt; wir schreiben $19^-$. April.
> Zweimal aber darf ein Datum im 19er-Zyklus nicht auftreten, kommt somit neben dem 19. April auch der 18. April vor, dann wird dieser zur Bestimmung des Ostertermins in den 17. April gewandelt; wir schreiben $18^-$. April.

Etwaige Doppelung des 17. gibt es dadurch nicht, weil die am Ende von Abschnitt 2 erstellten Aussagen auch weiterhin gelten. Der Sonderfall für den 18. tritt deshalb auch nur für $a \geq 11$ auf, d. h. in $8/19$ der Fälle. Fiktiv aber bleiben in den 30 OG der 19. und der 18. April ungeändert erhalten, bei Weiterschreiben der Tabelle für die OG durch Sonnen- oder Mondschaltung sind also für jedes feste $a$ (in jeder Spalte der Tabelle) der 17., 18., 19. April und danach der 21. März aufeinanderfolgend.

(Beim Weiterdenken der OG-Folgen beobachtet man ein leicht irrlichterndes Verhalten. So gilt z. B. in den 7 Säkularjahren 2200, 2300, ..., 2800 für die Schaltungen um $\Delta S - \Delta T$ der Reihe nach: $1-0=1$, $1-0=1$, $0-1=-1$, $1-0=1$, $1-0=1$, $1-1=0$, $0-0=0$.

Daher gelten ab 2400 (‚die $-1$') wieder die OG wie ab 2200, ab 2500 diejenigen ab 2300, ab 2600 um $1d$ erhöhte, die aber auch ab 2700 und ab 2800 unverändert bleiben bis 2899.)

> Die Ermittlung des Ostertermins für ein gewünschtes Jahr $X$ gelingt mit Ablesen oder Weiterschreiben der OG und Bestimmung des Sonntags danach; letztes mittels Sonntagsbuchstaben, oder mit Anzahl $S$ der Sonnenschaltungen und der Formel:
>
> der $(28-\mathrm{mod}(X+\mathrm{int}(X/4)-S,7))$-te März ist erster Sonntag nach dem 21. März

($\mathrm{int}(X/4)$ bezeichnet den ganzen Anteil von $X/4$). Die Formel ist Bestandteil der „revidierten Gaußschen Osterformel“ aus [9], man verifiziert sie leicht für ein festes (gegenwärtiges) Jahr, Induktionsschritte sind dann offensichtlich.

In der genannten Osterformel ist auch ein prägnant kurzer Term zur Realisierung der beiden soeben erwähnten Sonderbestimmungen enthalten, der Autor nennt sie dort „Alexandrinische Korrektur“. Gelehrte alexandrinische Geistliche haben, wie Joseph Bach in [1] ausführt, Bedeutsames in der Vorgeschichte zur Osterfestterminierung geleistet. In [8], S. 64 heißt es: „... insbesondere die beiden Sonderbestimmungen heißen in der neueren Literatur nach *Bach* – so bei *Zemanek* (hier [13], S. 53) –, da dieser sie formuliert, s. [1], S. 32 und S. 34f.“ Aber die Bestimmungen sind doch schon Jahrhunderte vor 1907 erlassen worden, sie sind auch von Gauss in [5] ausdrücklich berücksichtigt. Dessen Formel von 1800 erfaßte allerdings die Mondschaltungen ab 4200 nicht adäquat, die Verschiebung der Schaltung auf 4300, erstmals nach 400 Jahren, ist dort nicht realisiert, was er 1816 im Artikel [7] korrigierte.

## 5 Periodizitäten, Häufigkeiten der Osterdaten

Die Periode $\Delta_J$ der Osterdaten im Julianischen Kalender muß wegen der regelmäßigen Schaltjahre ein Vielfaches von 4 sein, und wegen Bestimmung der OG im 19jährigen Zyklus ein Vielfaches von 19, in Jahren ausgedrückt also $\Delta_J = k \cdot 4 \cdot 19$. In bezug auf die Wochentage muß es sich aber auch um ein Vielfaches von 7 handeln, d. h. $\Delta_J = k \cdot 4 \cdot 365,25 \cdot 19d = k \cdot 1461 \cdot 19d$, und weil hier sonst kein Faktor durch 7 teilbar ist, muß $k = 7$ sein für das kleinste $\Delta_J$, d. h., in Julianischen Jahren ist $\Delta_J = 7 \cdot 4 \cdot 19 = 532$.

Ostern tritt an einem bestimmten Datum des Zeitraums 22. März bis 25. April ein, wenn eine der für das Datum günstigen OG eintritt, also eine solche, die höchstens 7 Tage vor dem Datum liegt, und wenn zudem das Datum auf einen Sonntag fällt. Für die Daten 28. März bis 19. April kämen zwar jeweils 7 OG-Termine in Frage, aber aus dem Dionysischen Zyklus bestimmt man ihre jeweilige Anzahl $g$ wie folgt:

Für den
28., 31. März, 3., 5., 6., 8., 11., 14., 16., 19. April ist $g = 5$,
für den
29., 30. März, 1., 2., 4., 7., 9., 10., 12., 13., 15., 17., 18. April ist $g = 4$;

ferner: für den 26., 27. März, 20. April ist $g = 4$, für den 25. März, 21., 22. April ist $g = 3$,
für den 23., 24. März, 23., 24. April ist $g = 2$, für den 22. März, 25. April ist $g = 1$.

In der Zeitspanne von $\Delta_J$ Jahren tritt jede OG in $\dfrac{\Delta_J}{19}$ Jahren ein, denn der OG-Zyklus wird stets voll durchlaufen; die $g$ günstigen OG für ein Osterdatum treten in $\dfrac{\Delta_J}{19} \cdot g$ Jahren ein. In $\dfrac{1}{7}$ dieser Jahre fällt das Datum auf Sonntag, d. h. in $\dfrac{\Delta_J \cdot g}{19 \cdot 7} = 4 \cdot g$ Jahren, die relative Häufigkeit des Datums als Ostertermin ist $\dfrac{4 \cdot g}{\Delta_J} = \dfrac{g}{133}$.

> Die Osterdaten des Julianischen Kalenders sind (bezogen auf die Periode $\Delta_J$) auch im mittleren Abschnitt vom 28. März bis zum 19. April nicht konstant gleich häufig; dies resultiert aus den unterschiedlichen Anzahlen der günstigen Ostergrenzen für die Daten.

Wir bestimmen auch die Periode $\Delta_G$ der Osterdaten im Gregorianischen Kalender. Durch die Sonnen- und Mondschaltungen verschieben sich in $\Delta_G$ Jahren die OG um

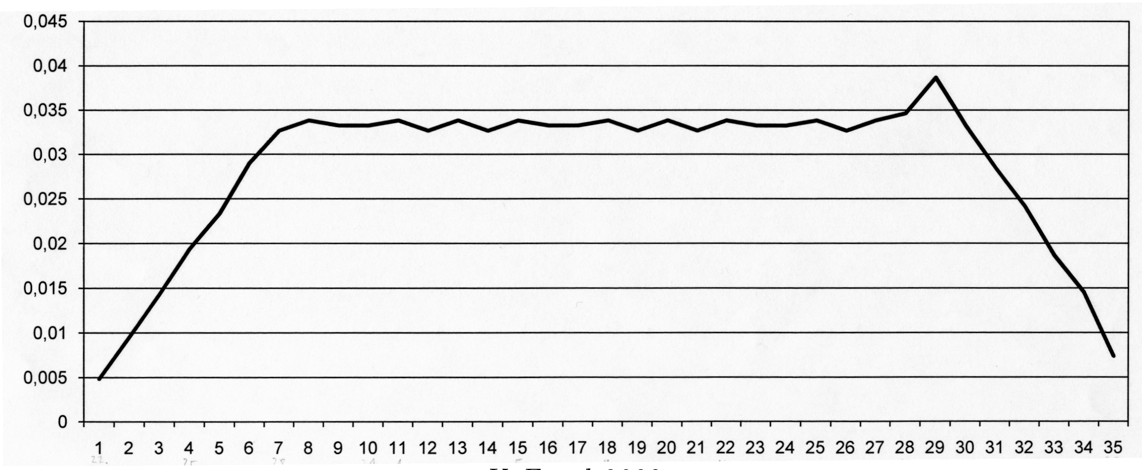
$$\frac{3d \cdot \Delta_G}{400} - \frac{8d \cdot \Delta_G}{2500} = \frac{43 \cdot \Delta_G}{400 \cdot 25} d\,.$$

Dies muß erstens ganz, zweitens Vielfaches von 30 sein wegen der 30 möglichen OG(0) (s. Abschnitt 4), und $\Delta_G$ muß wie schon beim Julianischen Kalender Vielfaches von 19 sein, in Jahren ausgedrückt somit $\Delta_G = k \cdot 400 \cdot 25 \cdot 30 \cdot 19$. In Tagen muß sich wiederum ein Vielfaches von 7 ergeben. Aber im Gregorianischen Kalender ist $400 \cdot 365,2425d = 146097d = 20871 \cdot 7d$, 400 Gregorianische Jahre bestehen genau aus 20871 vollen Wochen, es ist oben $k = 1$, und $\Delta_G = 400 \cdot 25 \cdot 30 \cdot 19$ Jahre $= 5.700.000$ Jahre.

Diese Zahl (angegeben auch in [9]) bleibe zunächst so stehen, und wir bestimmen relative Häufigkeiten der Ostertermine des Gregorianischen Kalenders in dieser Periode.

Während eines Vortrages im Jahre 2008 zeigte Herr K. Engel (Rostock) nachfolgendes Diagramm. Als Abszisse sind darin die 35 möglichen Osterdaten 22. März bis 25. April durch ihre *Festzahlen* von 1 bis 35 numeriert. Zum Nachsinnen über die Nichtkonstanz im mittleren Bereich wurde freundlich ermuntert, und wenn auch hierüber untereinander schon bald danach ein Austausch erfolgte, sei das Thema hier aufgegriffen.

**Häufigkeitsverteilung der Ostersonntage**
**über die Periode von 5.700.000 Jahren**




*K. Engel 2008*

Für jedes Datum vom 22. März bis zum 26. April bestimmt sich die Anzahl $g$ der günstigen OG aus der 30tägigen Zeitspanne 21M bis 19A (zunächst) simpel:

> Für den 28. März bis zum 17. April, und für den [18.], [19.], 20. April ist $g = 7$,
> für den 27. März, 21. April ist $g = 6$, für den 26. März, 22. April ist $g = 5$,
> für den 25. März, 23. April ist $g = 4$, für den 24. März, 24. April ist $g = 3$,
> für den 23. März, [25. April] ist $g = 2$, für den 22. März, [26. April] ist $g = 1$.

Aber nach der Alexandrinischen Korrektur – OG 19A wird zu 18A, 8/19 der OG 18A werden zu 17A – ergeben sich Veränderungen bei den eingeklammerten Daten:

> Für den 18. April wird $g = 7 + 8/19$, für den 19. April wird $g = 7 + 1 = 8$,
> für den 25. April wird $g = 2 - 8/19$, für den 26. April wird $g = 1 - 1 = 0$.

400 Gregorianische Jahre bilden, wie gesehen, eine Wochentagsperiode; die 400 Daten aber z. B. des 22. März bilden kein Vielfaches von 7, sie können demnach in 400 Jahren nicht gleich

oft auf jeden der 7 Wochentage fallen. Der 22. März hat eine Anzahl von Sonntagsterminen ($St$); seine Sa-, Fr-, ..., Mo-Anzahl sind aber gleichzeitig die $St$ je für den 23., 24., ..., 28. März, die $St$ sind somit nicht alle gleich. Man bestimmt für den 22. bis 28. März leicht die $St$-Folge 58, 57, 57, 58, 56, 58, 56, und diese setzt sich für die weiteren möglichen Ostertermine bis zum 25. April periodisch fort.

In der Zeitspanne von $\Delta_G$ Jahren tritt jede der 30 OG in $\dfrac{\Delta_G}{30}$ Jahren ein; die $g$ günstigen OG für ein Osterdatum treten in $\dfrac{\Delta_G}{30} \cdot g$ Jahren ein. Den etwas aufwendigen Nachweis hierfür sparen wir aus, zumal unsere Beurteilung von $\Delta_G$ noch mutieren wird. In $\dfrac{St}{400}$ dieser Jahre fällt das Datum auf Sonntag, d. h. in $\dfrac{\Delta_G \cdot g \cdot St}{30 \cdot 400} = 475 \cdot\ g \cdot St$ Jahren, die relative Häufigkeit des Datums als Ostertermin ist $\dfrac{g \cdot St}{12000}$.

Zum Vergleich mit dem obigen Diagramm notieren wir noch die Folge der $St$ für den Abschnitt vom 28. März bis zum 17. April mit konstantem $g = 7$ (Festzahlen 7 bis 27):

| $St$ | Festzahlen | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 58 | | 8 | | | 11 | | 13 | | 15 | | | 18 | | 20 | | 22 | | | 25 | | 27 |
| 57 | | | 9 | 10 | | | | | | 16 | 17 | | | | | | 23 | 24 | | | . |
| 56 | 7 | | | | | 12 | | 14 | | | | | 19 | | 21 | | | | | 26 | |

(dreimal die Folge 56, 58, 57, 57, 58, 56, 58)

Die herausfallenden Häufigkeiten des 18. und besonders des 19. Aprils (Festzahlen 28 bzw. 29) erklären sich aus $g \cdot St\,(18.\,\text{April}) = (7 + 8/19) \cdot 56 > 7 \cdot 58$ und $g \cdot St\,(19.\,\text{April}) = 8 \cdot 58$.

> Die Osterdaten des Gregorianischen Kalenders sind (bezogen auf die Periode $\Delta_G$) auch im mittleren Abschnitt vom 28. März bis zum 17. April nicht konstant gleich häufig; dies resultiert aus den unterschiedlichen Anzahlen der Sonntagstermine für die Daten.

## 6 Kalendertraditionen hinterfragt

Die Einführung des Gregorianischen Kalenders erfolgte in den christlichen Ländern zu recht unterschiedlichen Zeiten, auch hierzu findet man Ausführliches in [1] (speziell S. 20, 64). Im protestantischen Teil Deutschlands geschah dies am 1. März 1700, allerdings wurden bis 1775 noch nicht die zyklischen Gregorianischen OG übernommen, sondern die OG astronomisch bestimmt, wodurch 1724 und 1744 die Protestanten eine Woche vor den Katholiken Ostern feierten, was für 1778 und 1798 abermals drohte (computergestützt gelingt eine Verifikation). Am Schluß kommen wir auf diesen Zeitabschnitt noch zurück.

Die Ostkirchen (orthodoxe Kirchen, wie in Weißrußland, Rußland u. a.) bestimmen ihre christlichen Feste auch jetzt noch im Julianischen Kalender nach der ursprünglichen Festlegung durch Dionysius. Sie berücksichtigen keine Sonnen- und Mondschaltungen, es gelten die Dionysischen OG. Alle ihre Daten sind somit, wenn sie in unserem Kalender ausgedrückt werden, um $S$ Tage zu erhöhen (zur Zeit ist $S = 13$), dies gilt dann auch für Feiertage mit fixiertem Datum, wie Weihnachten. Die Ostertermine beider Kalender fallen bis 2099 entweder auf den (astronomisch) gleichen Tag, oder die Ostkirchen feiern Ostern eine Woche, vier Wochen oder fünf Wochen später (s. [2], dort auch alle Daten bis 2099, schon [1] enthält umfangreiche Angaben).

Beispielweise sind für 2013 (mit $a = 18$) die OG Gregorianisch 27M, Julianisch $17\text{A} + 13d = 30\text{A}$, und Ostersonntage sind der 31. März bzw. 5. Mai 2013 (5 Wochen später, schon deutlich gen Sommer gelegen); dagegen sind für 2014 (mit $a = 0$) die OG 14A bzw. $5\text{A} + 13d = 18\text{A}$, und Ostersonntag ist für beide Bestimmungen der 20. April 2014.

Irgendwie scheint es verlockend zu sein, mit großen Zeitspannen zu jonglieren.

Die Gregorianische OG-Folge wird durch ihre Schaltungen natürlich zu gewisser Zeit formal wieder genau auf die Dionysische fallen, und falls sie dann auch in den Wochentagen übereinstimmen, sind die Osterdaten in beiden Kalendern formal dieselben. Hierzu gibt es in den „Werken“ von Gauss Nachbemerkungen des Herausgebers Loewy im Artikel [7], daß dies „vom Jahre 6700 bis 6799“ sein wird, „wenn die beiden Kalender solange ungeändert im Gebrauch bleiben sollten. Einerlei Datum würde aber alsdann in beiden Kalendern immer um 7 volle Wochen auseinander sein.“ Und „diese Übereinstimmung tritt von 20700 bis 20799 von neuem ein, wo aber der Unterschied 22 Wochen betragen würde.“ (In der Tat ist z. B. $6700 = 1500 + 13 \cdot 400$, daher $S = 10 + 13 \cdot 3 = 49 = 7 \cdot 7$, und $6700 = 1800 + 2500 + 2400$, somit $T = 4 + 8 + 7 = 19$, $S - T = 49 - 19 = 30$; die ursprünglichen OG sind zwar Gregorianisch um $30d$ geändert, aber da sie ja stets mod 30 auf die Tage 21M bis $19^{-}$A reduziert wurden, wieder wie 532, jedoch sind die Gregorianischen Kalenderdaten für die Julianischen um $7 \cdot 7d$ zu erhöhen. (Im Jahrhundert ab 6700 würden so ‚Ostern und Pfingsten stets auf einen Tag fallen‘, Julianisch bzw. Gregorianisch.))

Der Herausgeber bezieht sich auch auf [6], wo Gauss mit seiner damals noch nicht durch [7] korrigierten Formel zu einer unrichtigen Angabe gekommen ist, aber Gauss schließt seine Formulierung bemerkenswert: „Es läßt sich übrigens zeigen, dass im 143. Jahrhundert, d. i. von 14200 bis 14299 Ostern in beiden Kalendern immer auf einerlei Datum fallen würde, wenn dieselben so lange ungeändert im Gebrauch bleiben sollten, was freilich nicht zu erwarten ist.“

Kehren wir nämlich zurück zu der vermeintlichen Periode $\Delta_G$ des Gregorianischen Osterkalenders von 5,7 Mill. Jahren. Das Kalenderjahr von $365,2425d$ ist zu groß für das astronomisch richtige Tropische Jahr von $365,2422d$, und zwar, wie man sieht, in 10.000 Jahren um $3d$. In 1 Mill. Jahren wären das schon $300d$, in 5,7 Mill. Jahren mehrere Jahre! Der ‚Frühlingspunkt' 21. März wäre in Richtung Sommer und mehrmals durch alle Jahreszeiten gedriftet!

> Die Periode von 5.700.000 Jahren für den (nicht modifizierten) Gregorianischen Osterkalender ist auch als mathematische Formalie kaum vertretbar, weil in solcher Zeitspanne das eigentliche Anliegen der Gregorianischen Reform grob verletzt würde.

Es wird nötig werden, die besagten $3d$ während 10.000 Jahren ausfallen zu lassen. Dazu gibt es von Zemanek den Vorschlag: „Der ‚ideale Kalender' könnte kein Schaltjahr in den Jahren 3200, 6400, 9600; 13200, 16400, 19600 usw. haben ... Man erkennt, wie präzise eingeregelt werden kann - mit nur sehr geringfügiger Änderung der gregorianischen Schaltregeln" ([13], S. 133, auch 34).

Unter diesem 10.000jährigen Zyklus der Sonnenschaltungen stellt sich vielleicht die Frage nach einer Periode $\Delta'_G$ neu. Analog zu $\Delta_G$ verschöben sich die OG in $\Delta'_G$ Jahren jetzt um

$$\frac{3d \cdot \Delta'_G}{400} + \frac{3d \cdot \Delta'_G}{10000} - \frac{8d \cdot \Delta'_G}{2500} = \frac{46 \cdot \Delta'_G}{10000} d = \frac{23 \cdot 2 \cdot \Delta'_G}{10000} d\,.$$

Dies müßte ganz, Vielfaches von 30, und $\Delta'_G$ Vielfaches von 19 sein, darum $\Delta'_G = k \cdot 10000 \cdot 15 \cdot 19$ Jahre $= k \cdot 3652422 \cdot 15 \cdot 19d$, Vielfaches von 7 erst für $k = 7$, es wäre $\Delta'_G = 19.950.000$ Jahre (der Länge $365,2422d$).

In derartiger Zeitspanne aber müßten auch die Mondschaltungen neu bedacht werden, denn in je 25.000 Jahren wären nicht $80d$, sondern $81d$ erforderlich (s. am Ende von Abschnitt 3). Hinzu käme, daß die verwendete Länge des Tropischen Jahres immer noch eine Rundung ist von $365,24219879d$. Man muß sich wohl auch hüten, von Konstanz der verwendeten Parameter auszugehen.

> Die Suche nach einer glaubhaften Periode bleibt im gegenwärtigen wie auch im modifizierten Osterkalender vergeblich.

## 7 Ergänzende Rück- und Ausblicke

Der astronomische Frühlingsanfang fällt nicht konstant auf den 21. März, was schon ein Blick in den Taschenkalender belegt. In der jüngeren Vergangenheit (seit 1980) war er in Jahren, die bei Division durch 4 den Rest 0, 1 oder 2 lassen, bereits am 20. März, nur beim Rest 3 am 21. Normaljahre von $365d$ sind nämlich um $0,2422d$ (fast $5h\,49min$) zu kurz, um diese Zeit (etwa) verspätet sich der Termin von Jahr zu Jahr, mit einem Schalttag in vier Jahren aber wird $4 \cdot 0{,}2422d - 1d = -0{,}0312d$ (beinahe $-45min$), der Termin tritt also ca. $45min$ eher ein als vier Jahre zuvor. Mit dem Jahr 2011 (Rest 3) endete so der genannte Ablauf, 2015 wird schon am 20. März Frühlingsanfang sein. Ab 2012 liegt er daher in diesem Jahrhundert stets vor dem 21. März, in Jahren des Restes 0, und dann auch 1, späterhin sogar am 19.

Pauschale Überlegungen ergeben: 400 Jahre nach 1903 ist Frühlingsanfang im Jahre 2303 wegen $-0{,}0312d \cdot 25 \cdot 4 + 3d = -0{,}12d$ ca. $0{,}12d$ ($2h\,50min$) eher; weiterhin aber liegt 2303 der späteste aller Termine nach 1903, und folglich 1903 ebenso derjenige ab 1583 (nach 1503). Nun ist $2012 = 1903 + (25 + 2) \cdot 4 + 1$, grob überschlagen gilt also für den Unterschied der Frühlingsanfangstermine von 1903 und (dem Schaltjahr) 2012

$$27 \cdot (-0{,}0312d) + (0{,}2422d - 1d) = -1{,}6002d > -(1d\,15h)\,,$$

somit war 1903 weniger als $1d\,15h$ später Frühlingsanfang als 2012, und dieser war am 20. März um 6:12 MEZ, d. h.:

> Während des Gregorianischen Kalenders lag im Jahre 1903 der späteste Termin für den astronomischen Frühlingsanfang, und zwar in MEZ am 21. März.
>
> Oftmals aber war astronomischer Frühlingsanfang bereits am 20. März, er wird auch auf den 19. fallen (2096 hat den frühesten Termin für dieses Jahrhundert – und damit für die drei folgenden). Bis zu einer ergänzenden Schalttagskorrektur werden sich die Termine in der Tendenz kalendarisch weiter verfrühen.

Der astronomische Frühlingsvollmond kann somit durchaus auch vor dem 21. März eintreten. 2076 werden Frühlingsanfang am 19., Vollmond am 20. März sein; hingegen gilt wegen $\mathrm{mod}\,(2076, 19) = 5$ als OG der $19^{-}$. April, also 18. April, ein Sonnabend, Ostern wird erst am 19. April sein, und nicht bereits vier Wochen eher am 22. März.

Solche abweichenden Gegebenheiten, zumal der MEZ-Zone, sind für den Gregorianischen Osterkalender irrelevant. In ihm gelten Ostergrenzen mit Daten ab dem 21. März weltweit, und diese sind gesteuert durch Sonnen- und Mondschaltungen, wie sie bereits bei Einführung des Kalenders beeindruckend weitsichtig vorgesehen wurden. Der Autor von [9] spricht zu Recht schon im Untertitel von einem „wissenschaftlichen Meisterwerk der späten Renaissance", dann auch von einem „Kulturgut der Menschheit", und plädiert leidenschaftlich für dessen Schutz. Nicht einleuchtend ist mir darum sein dann doch geäußerter Vorschlag, die Schaltungen der OG zu modifizieren, und zwar auf $13d$ in 3.000 Jahren (um dem oben in Abschnitt 4 angedeuteten Irrlichtern gegenzusteuern). Durchschnittlich entspräche das $43\,{}^{1}\!/\!{}_{3}d$ in 10.000 Jahren, aber im Gregorianischen Kalender (Abschnitt 5, bei $\Delta_G$) sind es $43d$, und bei $3d$ zusätzlicher Schaltung (Abschnitt 6, bei $\Delta'_G$) $46d$ je in 10.000 Jahren, langfristig würden beide Werte verfehlt.

Mehr als 100 Jahre älter noch als der Gregorianische Kalender ist die Astronomische Uhr in St.-Marien aus den Jahren um 1472 ([10], S. 24). Auf ihr bewegen sich u. a. Sonnen- und Mondsymbol über einen geschnitzten Zodiakus, die Übersetzungsverhältnisse der Antriebsräder ergeben Näherungswerte für das Jahr bzw. den synodischen Monat, und es wurden schon damals im Uhrwerk Möglichkeiten für korrigierende Justierungen vorgesehen.


*Astronomische Uhr in St.-Marien, 7.12.2009 14 Uhr Foto M. Berger*

Solche erfolgten jüngst am 9. Juli 2012, und sie bewirken, daß für die nächsten 4 Jahre wieder die durchschnittlichen Abweichungen zwischen dem (regelmäßigen) Uhrmond und dem (nur in langem Durchschnitt regelmäßigen) Naturmond so gering ausfallen, wie im Rahmen der verfügbaren Zahnradkonstellationen möglich (vgl. Abschnitt 3).

Joseph Bach [1] formuliert, „wie wichtig die richtige Bestimmung des Osterdatums für das kirchliche und bürgerliche Leben sowie für die chronologische Festlegung historischer Tatsachen ist. Ostern ist so zu sagen der geometrische Ort, von dem aus die für die Historiker oft so hochwichtigen Datumsprobleme erschlossen werden können.“ Hierzu referieren wir abschließend ein Beispiel aus dem sakralen Umfeld unserer Betrachtungen.

Johann Sebastian Bach hat in Leipzig die – von uns besonders geschätzte – Kantate 140, „Wachet auf, ruft uns die Stimme“, für den 27. Sonntag nach Trinitatis komponiert. Das nicht überlieferte Entstehungsjahr der Kantate konnte später bestimmt werden unter Beantwortung der Frage, wann es denn den genannten Sonntag, der damals nur relativ selten vorkam, zu Bachs Leipziger Wirkungszeit überhaupt gegeben hat. Dazu mußte Ostern nämlich vor dem 27. März, die OG also vor dem 26. März gelegen haben, relevant sind die Jahre 1723 bis 1749. Man findet mit der Tabelle von Abschnitt 4: OG 22M ($a = 2$) für 1731, OG 24M ($a = 10$) 1739, OG 21M ($a = 13$) 1742 und 1723. Sonntagsdaten für die genannten Jahre hatten wir bereits in Abschnitt 1 bestimmt, und erhalten die Ostertermine: 25. März 1731, 29. März 1739, 25. März 1742 und 28. März 1723. Es bleiben möglich die Jahre 1731 und 1742. Unter diesen sprechen nach Smend ([12], S. 42) überzeugende Merkmale der Handschrift und des Stils für die Komposition im Jahr 1731 (zum 25. November).

Zu bedenken wäre aber noch die schon erwähnte Situation, daß im protestantischen Leipzig der Ostertermin 1724 und 1744 vom Gregorianischen Kalender abgewichen ist. Für 1724 und 1744 mit $a = 14$ bzw. 15 findet man als OG den 9. April bzw. 29. März, beide Daten bestimmt man als Sonntage, folglich als die protestantischen Ostersonntage, weil Ostern gregorianisch jeweils eine Woche später sein mußte. Den 27. Sonntag nach Trinitatis aber konnte es in beiden Jahren nicht geben.

Heute gilt eine dahingehend geänderte evangelische Ordnung, daß es immer als solchen den „Letzten Sonntag im Kirchenjahr“ gibt, der aber inhaltlich dem 27. nach Trinitatis entspricht (darum wird im Rundfunk auch in jedem Jahr die Kantate 140 gesendet). Ebenso gibt es den „Vorletzten“ und „Drittletzten Sonntag im Kirchenjahr“ stets, die Numerierung davor geht höchstens bis zum 24. Sonntag nach Trinitatis, und dieser ist so der seltenste.

Frühe Ostertermine sind auf der neuen Kalenderscheibe rar. Der 22. März kann nicht vorkommen, weil in diesem und dem folgenden Jahrhundert die frühest mögliche OG 21M fehlt, aber auch der 23. und 24. März treten nicht auf. Ostern am 25. März gibt es dreimal, zuerst 2035, auch als Termin vor dem 27. März. Dann wird es wieder den 24. Sonntag nach Trinitatis geben, erstmals seit 2008 und davor im bewegenden Herbst 1989.

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# On the Solutions of Two-Scale Difference Equations

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ABSTRACT. This paper deals with specific two-scale difference equations which are equivalent to a system of functional equations. Such equations have a continuous solution if the coefficients $c_j$ of the corresponding characteristic polynomial $P$ satisfy condition $|c_j| < 1$ for all $j$. By means of some functional relations for the solution we show that it is Hölder continuous and we determine the optimal Hölder exponent. Moreover we give a condition which is necessary and sufficient for the differentiability almost everywhere where we apply Borel's normal number theorem. If the coefficients $c_j$ are nonnegative then the solution is a singular function. Special cases are the well-known singular functions of *de Rham* and of *Cantor*.

## 1 Introduction

A *two-scale difference equation* (*dilation equation*) is a functional equation of the form

$$\varphi\left(\frac{x}{d}\right) = \sum_{j=0}^{p-1} c_j \varphi(x-j) \tag{1.1}$$

with dilation parameter $d > 1$ and complex coefficients $c_j$ where $c_0 c_{p-1} \neq 0$, $p \geq 2$. Such equations especially with $d = 2$ appear in wavelet theory and in subdivision schemes where nontrivial compactly supported Lebesgue-integrable solutions are demanded, cf. [5], [7], [8], [9].

In this paper we consider the two-scale difference equation (1.1) with $d = p$, that means

$$\varphi\left(\frac{x}{p}\right) = \sum_{j=0}^{p-1} c_j \varphi(x-j) \qquad (x \in \mathbb{R}) \tag{1.2}$$

under the condition

$$\sum_{j=0}^{p-1} c_j = 1 \tag{1.3}$$

and we are interested to solutions $\varphi$ which satisfy the boundary conditions

$$\varphi(x) = 0 \quad \text{for} \quad x < 0, \qquad \varphi(x) = 1 \quad \text{for} \quad x > 1. \tag{1.4}$$

It is easy to see that under these conditions equation (1.2) with (1.3) can be written as system of functional equations. Replacing $x$ in (1.2) by $k + x$ with $k \in \{0, 1, \ldots, p-1\}$ and $x \in [0, 1]$ we get in view of (1.4) the following system of equations

$$\varphi\left(\frac{k+x}{p}\right) = b_k + c_k \varphi(x) \qquad (0 \le x \le 1) \tag{1.5}$$

with

$$b_k = \sum_{j=0}^{k-1} c_j \tag{1.6}$$

$k = 0, 1, \ldots, p-1$, cf. [18]. Such systems of equations are intensively investigated by R. Girgensohn, see [14], [15], [16]. If $|c_j| < 1$ for all $j = 0, 1, \ldots, p-1$ then there exists exactly one bounded $\varphi : [0, 1] \mapsto \mathbb{R}$ which satisfies (1.5) with (1.6) and (1.3). This function $\varphi$ is continuous and given in terms of the $p$-adic expansion of $x$ by

$$\varphi\left(\sum_{n=1}^{\infty} \frac{\xi_n}{p^n}\right) = \sum_{n=0}^{\infty} b_{\xi_n} \prod_{k=1}^{n-1} c_{\xi_k}, \tag{1.7}$$

cf. [14], see also [18, Theorem 2]. In particular, $\varphi(0) = 0$ and $\varphi(1) = 1$ so that $\varphi$ can be extended by (1.4) to $x \in \mathbb{R}$, and this extended function is a continuous solution of (1.2) and satisfies (1.4). In this sense the two-scale difference equation (1.2) with (1.3) is equivalent to the system of equations (1.5) with (1.6) and (1.3).

The polynomial

$$P(z) = \sum_{j=0}^{p-1} c_j z^j \tag{1.8}$$

with $P(0) \neq 0$ and $P(1) = 1$ is called the *characteristic polynomial* of the equation (1.2). Simple examples are the extended functions of *de Rham* and of *Cantor*.

**1.** (*De Rham's function*) In case $P(z) = a + (1-a)z$ with $a \in (0, 1)$ equation (1.2) reads

$$\varphi\left(\frac{x}{2}\right) = a\varphi(x) + (1-a)\varphi(x-1) \qquad (x \in \mathbb{R}) \tag{1.9}$$

which in view of (1.4) can be written as system of functional equations

$$\varphi\left(\frac{x}{2}\right) = a\varphi(x), \qquad \varphi\left(\frac{x+1}{2}\right) = a + (1-a)\varphi(x) \tag{1.10}$$

with $0 \le x \le 1$ and de Rham's function is the uniquely bounded solution, cf. e.g. [18].

**2.** (*Cantor's function*) In case $P(z) = (1 + z^2)/2$ equation (1.2) reads

$$\varphi\left(\frac{x}{3}\right) = \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(x-2) \qquad (x \in \mathbb{R}). \tag{1.11}$$

In view of (1.4) this equation can be written as system of equations

$$\varphi\left(\frac{x}{3}\right) = \frac{1}{2}\varphi(x), \qquad \varphi\left(\frac{x+1}{3}\right) = \frac{1}{2}, \qquad \varphi\left(\frac{x+2}{3}\right) = \frac{1}{2} + \frac{1}{2}\varphi(x) \tag{1.12}$$

with $0 \le x \le 1$, and Cantor's function is the unique bounded solution of this system, cf. [21], (see also [20], p. 241).

In case $c_j \ge 0$ for all $j = 0, 1, \dots, p-1$ one can interpret the coefficients as probabilities $p_j = c_j$ and the solution $\varphi$ as a distribution function which is a measure-preserving mapping, cf. [4, Section 3]. The figure on p. 37 in [4] shows the graph of $\varphi$ in case $p = 2$, $p_0 = 0,7$ and $p_1 = 0,3$ ($\varphi$ is de Rham's function with respect to the parameter $a = 0,7$).

According to (1.4) we are only interested to the solution $\varphi$ of (1.2) in $[0, 1]$. We always assume that $|c_j| < 1$ for all $j = 0, 1, \dots, p-1$ which guarantees the existence of a continuous solution $\varphi$ with $\varphi(0) = 0$ and $\varphi(1) = 1$. In the simple case $c_j = \frac{1}{p}$ for all $j$ we have $\varphi(x) = x$ for $x \in [0, 1]$. In the following we always exclude this trivial case. We show in this paper that the solution $\varphi$ of (1.2) with (1.3), (1.4) satisfies some functional relations (Proposition 2.3) and that it has in $[0, 1]$ the following properties:

1. If $c_j \ge 0$ for all $j$ then $\varphi$ is an increasing function (Proposition 2.5).
2. If not $c_j \ge 0$ for all $j$ then in no nonempty subinterval of $[0, 1]$ $\varphi$ has finite variation (Proposition 2.6).
3. If $|c_j| < 1$ for all $j$ then $\varphi$ is Hölder continuous, i.e.

   $$|\varphi(x) - \varphi(y)| \le A|x - y|^\alpha$$

   with the optimal Hölder exponent $\alpha = \min\{-\log_p |c_0|, \dots, -\log_p |c_{p-1}|\}$ and coefficient $A$ with $1 \le A \le p^{1-\alpha}\frac{p-1}{p^\alpha - 1}$ (Theorem 3.6).
4. If $\varphi$ is differentiable at the point $x_0$ then $\varphi'(x_0) = 0$ (Proposition 4.2).
5. If $\min|c_j| \ge \frac{1}{p}$ then $\varphi$ is nowhere differentiable in [0,1], and if $\min|c_j| < \frac{1}{p}$ then both sets, where $\varphi$ is differentiable and where $\varphi$ is not differentiable have positive Hausdorff dimension (Theorem 4.11).
6. If $p\,M_0 < 1$, where $M_0 = |c_0 c_1 \cdots c_{p-1}|^{1/p}$, then $\varphi$ is differentiable almost everywhere and if $p\,M_0 \ge 1$ then it is almost nowhere differentiable (Theorem 4.12).
7. If $0 \le c_j < 1$ and $\min c_j = 0$ then $\varphi$ is constant on the components $J_{m,n}$ of an open set $G \subseteq [0, 1]$ with Lebesgue measure $|G| = 1$. These intervals can be represented by means of a sequence $\gamma_n$ (Theorem 5.4, Example 5.6).

**Example 1.1** For $0 < a < 1$ the equation

$$\varphi\left(\frac{x}{3}\right) = a\varphi(x) + (1-2a)\varphi(x-1) + a\varphi(x-2) \qquad (x \in \mathbb{R}) \tag{1.13}$$

has a continuous solution $\varphi$ satisfying (1.4). For $0 \le x \le 1$ we have: $\varphi$ is increasing for $0 < a < \frac{1}{2}$, $\varphi$ is Cantor's function for $a = \frac{1}{2}$ and $\varphi$ does not have finite variation for $\frac{1}{2} < a < 1$. Further, $\varphi$ is nowhere differentiable for $\frac{2}{3} \le a < 1$. If $a_0$ is the positive solution of $27a^2(2a-1) = 1$, i.e. $a_0 = 0,5592\ldots$, then $\varphi$ is differentiable almost everywhere for $0 < a < a_0$ and almost nowhere differentiable for $a_0 \le a < 1$. So it is astonishing that in case $\frac{1}{2} < a < a_0$ the continuous solution $\varphi$ does not have finite total variation though the derivative vanishes almost everywhere.

**Remark 1.2** **1.** Hölder continuity of compactly supported solutions $\varphi$ of (1.1) are intensive investigated, e.g. for the Hölder exponent there are bounds in terms of the joint spectral radius of two matrices determined of the coefficients $c_j$, cf. [5, Theorem 4.3], [6], [7].

**2.** The optimal Hölder exponent $\alpha = \log_3 2$ of Cantor's function is already known from [17].

**3.** The optimal Hölder exponent $\alpha = \min\{-\log_2 a, -\log_2(1-a)\}$ of de Rham's function was already determined in [2, Section 2]. Remark 2 and Figure 3 in [2] show a comparison with the Hölder exponent $\mu = -\frac{1}{\log 4}\log(2a^2 - 2a + 1)$ obtainable by means of the corresponding joint spectral radius, cf. [6].

## 2 Functional relations

We start with a replicative relation, cf. [18].

**Proposition 2.1** *The solution $\varphi$ of system (1.5), (1.6) satisfies the replicative relation*

$$\sum_{k=0}^{p-1} \varphi\left(\frac{k+x}{p}\right) = \varphi(x) + C \qquad (x \in [0,1]) \tag{2.1}$$

*with the constant*

$$C = p - 1 - P'(1). \tag{2.2}$$

**Proof:** Equation (2.1) follows from (1.5) by summation where $x = 0$ yields for the constant in (2.1)

$$C = \sum_{k=1}^{p-1} \varphi\left(\frac{k}{p}\right).$$

From (1.5) and (1.6) we get

$$\varphi\left(\frac{k}{p}\right) = \sum_{j=0}^{k-1} c_j$$

so that

$$\begin{aligned} C &= (p-1)c_0 + (p-2)c_1 + \ldots + c_{p-2} \\ &= (p-1)(c_0 + \ldots + c_{p-1}) - \{c_1 + 2c_2 + \ldots + (p-1)c_{p-1}\} \\ &= (p-1)P(1) - P'(1). \end{aligned}$$

In view of $P(1) = 1$ it follows (2.2). □

In order to derive further functional relations for the solutions of (1.2) we introduce a sequence $C_k(c)$ depending on an arbitrary parameter $c \neq 0$ as follows: For $j \in \{0, 1, \ldots, p-1\}$ we put $C_j(c) = \frac{c_j}{c}$ where $c_j$ are the coefficients of (1.2) and in general by the recursion:

$$C_{kp+j}(c) = C_k(c)C_j(c) \qquad (k \geq 1, j \in \{0, 1, \ldots, p-1\}). \tag{2.3}$$

Obviously, if $k$ has the $p$-adic representation

$$k = \sum_{\nu=0}^{n} k_\nu p^\nu, \qquad (k_\nu \in \{0, 1, \ldots, p-1\}) \tag{2.4}$$

then we have the explicit representation

$$C_k(c) = \prod_{j=0}^{p-1} \left(\frac{c_j}{c}\right)^{s_j(k)} \tag{2.5}$$

where $s_j(k)$ denotes the total number of occurrences of the digit $j$ in the $p$-adic expansion (2.4) of $k$.

**Remark 2.2** We use the parameter $c$ in two cases:

**1.** In case $c = c_0$ we have $C_0(c_0) = 1$ and from (2.3) it is easy to see that the numbers $C_k := C_k(c_0)$ have the generating function

$$G(z) := \prod_{j=0}^{\infty} \frac{1}{c_0} P\left(z^{p^j}\right) = \sum_{k=0}^{\infty} C_k z^k \tag{2.6}$$

which converges for $|z| < 1$. Let us mention that the unit circle is a natural bound of convergence for $G$, cf. [12].

**2.** In Section 3 (Hölder continuity) we put $c = \max\{|c_0|, \ldots, |c_{p-1}|\}$ and so we are able to estimate the Hölder coefficient.

In the following we need the function

$$\varphi^*(x) = 1 - \varphi(1-x) \qquad (x \in \mathbb{R}) \tag{2.7}$$

which is the solution of the reversed two-scale difference equation

$$\varphi^*\left(\frac{x}{p}\right) = \sum_{j=0}^{p-1} c_j^* \varphi^*(x-j) \qquad (x \in \mathbb{R}). \tag{2.8}$$

where

$$c_j^* = c_{p-1-j} \tag{2.9}$$

cf. [3].

**Proposition 2.3** *The solution $\varphi$ of system (1.5) satisfies the functional equations*

$$\varphi\left(\frac{k+t}{p^n}\right) = \varphi\left(\frac{k}{p^n}\right) + c^n C_k(c)\varphi(t) \quad (0 \le t \le 1) \tag{2.10}$$

*where $n \in \mathbb{N}$, $k = 0, 1, \ldots, p^n - 1$, $C_k(c)$ from (2.5) with*

$$s_0(k) + \ldots + s_{p-1}(k) = n \tag{2.11}$$

*and*

$$\varphi\left(\frac{k-t}{p^n}\right) = \varphi\left(\frac{k}{p^n}\right) - c^n C_{k-1}(c)\varphi^*(t) \quad (0 \le t \le 1) \tag{2.12}$$

*for $k = 1, 2, \ldots, p^n$ with $\varphi^*$ from (2.7). Moreover*

$$\varphi\left(\frac{k}{p^n}\right) = c^n \sum_{j=0}^{k-1} C_j(c). \tag{2.13}$$

**Proof:** We prove (2.10) by induction on $n$. For $n = 1$ the equations (2.10) are equivalent to the system (1.5). If (2.10) with (2.11) in $C_k(c)$ holds for a fixed $n$ then for $\frac{j+t}{p}$ instead of $t$ with $j \in \{0, 1, \ldots, p-1\}$ and $0 \le t \le 1$ we have

$$\begin{aligned} \varphi\left(\frac{kp+j+t}{p^{n+1}}\right) &= \varphi\left(\frac{k}{p^n}\right) + c^n C_k(c)\varphi\left(\frac{j+t}{p}\right) \\ &= \varphi\left(\frac{k}{p^n}\right) + c^n C_k(c)\varphi\left(\frac{j}{p}\right) + c^{n+1} C_k(c)\frac{c_j}{c}\varphi(t). \end{aligned}$$

For $t = 0$ it follows

$$\varphi\left(\frac{kp+j}{p^{n+1}}\right) = \varphi\left(\frac{k}{p^n}\right) + c^n C_k(c)\varphi\left(\frac{j}{p}\right)$$

and hence we get in view of (2.3)

$$\varphi\left(\frac{kp+j+t}{p^{n+1}}\right) = \varphi\left(\frac{kp+j}{p^{n+1}}\right) + c^{n+1} C_{kp+j}(c)\varphi(t)$$

with $s_0(kp+j)+\ldots+s_{p-1}(kp+j)=n+1$. Thus (2.10) with (2.11) in $C_k(c)$ is proved by induction. Now (2.13) follows from (2.10) for $t=1$ and $\varphi(1)=1$ by summation. Equation (2.10) with $k-1$ instead of $k$ and $1-t$ instead of $t$ yields in view of (2.7)

$$\begin{aligned}\varphi\left(\frac{k-t}{p^n}\right) &= \varphi\left(\frac{k-1}{p^n}\right)+c^nC_{k-1}(c)\varphi(1-t)\\ &= \varphi\left(\frac{k-1}{p^n}\right)+c^nC_{k-1}(c)-c^nC_{k-1}(c)\varphi^*(t).\end{aligned}$$

For $t=0$ it follows

$$\varphi\left(\frac{k}{p^n}\right)=\varphi\left(\frac{k-1}{p^n}\right)+c^nC_{k-1}(c)$$

and hence (2.12). □

**Example 2.4** (*De Rham's function*) In case $P(z)=a+(1-a)z$ we have $p=2$ and equation (1.9), i.e. $c_0=a$, $c_1=1-a$. For $c=a$ we have by (2.5) that $C_k=C_k(a)=q^{s_1(k)}$ with $q=\frac{1-a}{a}$ where $s_1(k)$ denotes the number of ones in the dyadic representation of $k$, and the generating function (2.6) reads

$$G(z)=\prod_{j=0}^{\infty}\left(1+qz^{2^j}\right)=\sum_{k=0}^{\infty}q^{s_1(k)}z^k. \tag{2.14}$$

Formulas (2.10) and (2.13) yield the known relations

$$\varphi\left(\frac{k+t}{2^n}\right)=\varphi\left(\frac{k}{2^n}\right)+a^nq^{s_1(k)}\varphi(t)\quad(0\le t\le 1)$$

and

$$\varphi\left(\frac{k}{2^n}\right)=a^n\sum_{j=0}^{k-1}q^{s_1(j)}$$

for de Rham's function $\varphi$, cf. [1].

**Proposition 2.5** *In case $c_j\ge 0$ for all $j=0,1,\ldots,p-1$ the solution $\varphi$ of (1.5) is an increasing function, and in case $c_j>0$ it is strictly increasing.*

**Proof:** If $c_j\ge 0$ for all $j$ then we have $0\le c_j<1$ since $c_0>0$, $c_{p-1}>0$ and (1.3). Hence the solution $\varphi$ is continuous. From (2.10) we get for $n\in\mathbb{N}$ and $k=0,1,\ldots,p^n-1$

$$\varphi\left(\frac{k+1}{p^n}\right)\ge\varphi\left(\frac{k}{p^n}\right)$$

so that the continuous function $\varphi$ is increasing. In case $c_j>0$ for all $j$ equation (2.10) implies

$$\varphi\left(\frac{k+1}{p^n}\right)>\varphi\left(\frac{k}{p^n}\right)$$

so that indeed $\varphi$ is strictly increasing in $[0,1]$. □

**Proposition 2.6** *If not $c_j \geq 0$ for all $j = 0, 1, \ldots, p-1$ then in no nonempty subinterval of $[0, 1]$ the solution $\varphi$ of (1.2) has finite total variation.*

**Proof:** If not $c_j \geq 0$ for all $j$ then owing to (1.3) we have $|c_0| + \ldots + |c_{p-1}| > 1$. From (1.5) we get for $k \in \{0, \ldots, p-1\}$

$$\varphi\left(\frac{k+1}{p}\right) - \varphi\left(\frac{k}{p}\right) = c_k$$

and hence

$$\sum_{k=0}^{p-1} \left|\varphi\left(\frac{k+1}{p}\right) - \varphi\left(\frac{k}{p}\right)\right| = \sum_{k=0}^{p-1} |c_k|$$

and by induction on $n$

$$\sum_{k=0}^{p^n-1} \left|\varphi\left(\frac{k+1}{p^n}\right) - \varphi\left(\frac{k}{p^n}\right)\right| = \left(\sum_{k=0}^{p-1} |c_k|\right)^n .$$

In view of $|c_0| + |c_1| + \ldots + |c_{p-1}| > 1$ it follows that $\varphi$ does not have finite total variation in $[0, 1]$. From (2.10) we conclude that this is valid also for the intervals $[\frac{k}{p^n}, \frac{k+1}{p^n}]$ with $n \in \mathbb{N}$ and $k = 0, 1, \ldots, p^n - 1$. $\square$

## 3 Hölder continuity

We assume that $|c_j| < 1$ for all $j = 0, 1, \ldots, p-1$ so that the solution $\varphi$ of (1.2) is continuous. In order to verify the Hölder continuity of $\varphi$ we introduce the notation

$$S_k(c) := \sum_{j=0}^{k-1} C_j(c) \tag{3.1}$$

for the sum in (2.13), i.e. we have

$$\varphi\left(\frac{k}{p^n}\right) = c^n S_k(c). \tag{3.2}$$

**Lemma 3.1** *The sequence $S_k(c)$ has following properties:*

**(i)** $S_{pk}(c) = \frac{1}{c} S_k(c) \quad (k \geq 1)$.

**(ii)** $S_{p^n}(c) = \frac{1}{c^n} \quad (n \geq 0)$.

**(iii)** $S_{kp^n+\ell}(c) = S_{p^n}(c) S_k(c) + C_k(c) S_\ell(c) \qquad (0 \leq k < p, \quad n \geq 1, \quad 0 \leq \ell < p^n)$.

**Proof:** **(i)** For given $k \geq 1$ we choose $n$ such that $k < p^{n-1}$. From (2.13) and (3.1) we get

$$S_{pk}(c) = \frac{1}{c^n}\varphi\left(\frac{pk}{p^n}\right) = \frac{1}{c^n}\varphi\left(\frac{k}{p^{n-1}}\right) = \frac{1}{c}S_k(c)$$

which implies **(i)**.

**(ii)** follows from (2.13) and $\varphi(1) = 1$.

**(iii)** From (2.10) and (2.13) we get

$$\varphi\left(\frac{k + \frac{\ell}{p^n}}{p}\right) = \varphi\left(\frac{k}{p}\right) + c\,C_k(c)\varphi\left(\frac{\ell}{p^n}\right) = c\,S_k(c) + c^{n+1}C_k(c)S_\ell(c).$$

On the other side we have

$$\varphi\left(\frac{kp^n + \ell}{p^{n+1}}\right) = c^{n+1}S_{kp^n+\ell}(c)$$

and in view of **(ii)** it follows **(iii)**. □

Now we choose the parameter $c = \mathbf{c}$ where

$$\mathbf{c} := \max\{|c_0|, |c_1|, \ldots, |c_{p-1}|\}, \tag{3.3}$$

cf. Remark 2.2. Then $|C_k(\mathbf{c})| \leq 1$ for $k \in \{0, 1, \ldots, p-1\}$ and (2.3) implies

$$|C_k(\mathbf{c})| \leq 1 \qquad (k \in \mathbb{N}_0). \tag{3.4}$$

In view of (1.3) we have $\frac{1}{p} \leq \mathbf{c} < 1$. In case $\mathbf{c} = \frac{1}{p}$ we have $c_j = \frac{1}{p}$ for all $j = 0, 1, \ldots, p-1$ and $\varphi(x) = x$ for $0 \leq x \leq 1$. If we exclude this trivial case then

$$\frac{1}{p} < \mathbf{c} < 1. \tag{3.5}$$

For the parameter $\mathbf{c}$ from (3.3) satisfying (3.5) we put

$$\alpha := -\log_p \mathbf{c}, \tag{3.6}$$

i.e.

$$\mathbf{c}\,p^\alpha = 1 \tag{3.7}$$

and (3.5) implies

$$0 < \alpha < 1. \tag{3.8}$$

**Lemma 3.2** *With $\alpha$ from (3.6) and $\mathbf{c}$ from (3.3) the sequence $\frac{1}{k^\alpha}S_k(\mathbf{c})$ is bounded. More precisely, for*

$$K := \sup_k \left\{\left|\frac{1}{k^\alpha}S_k(\mathbf{c})\right|\right\} \tag{3.9}$$

*we have the estimate*

$$1 \leq K \leq \frac{p-1}{p^\alpha - 1}. \tag{3.10}$$

**Proof:** According to Lemma 3.1/**(ii)** and (3.7) we have

$$\frac{1}{p^\alpha} S_p(\mathbf{c}) = 1$$

and hence $K \geq 1$. Moreover, by Lemma 3.1/**(i)** and (3.7)

$$\frac{1}{(pk)^\alpha} S_{pk}(\mathbf{c}) = \frac{1}{k^\alpha} S_k(\mathbf{c}) \tag{3.11}$$

so that

$$\sup_k \left| \frac{1}{k^\alpha} S_k(\mathbf{c}) \right| = \limsup_{k\to\infty} \left| \frac{1}{k^\alpha} S_k(\mathbf{c}) \right|. \tag{3.12}$$

For integer $n \geq 1$ let be

$$K_n := \max\left\{ \left| \frac{1}{k^\alpha} S_k(\mathbf{c}) \right| : p^{n-1} \leq k \leq p^n - 1 \right\}$$

then by (3.11) we have $K_n \leq K_{n+1}$.

Owing to Lemma 3.1/**(iii)** and to (3.11) we have

$$\frac{1}{(kp^n+\ell)^\alpha} S_{kp^n+\ell}(\mathbf{c}) = \left( \frac{kp^n}{kp^n+\ell} \right)^\alpha \frac{1}{k^\alpha} S_k(\mathbf{c}) + \left( \frac{\ell}{kp^n+\ell} \right)^\alpha \frac{C_k(\mathbf{c})}{\ell^\alpha} S_\ell(\mathbf{c}) \tag{3.13}$$

for $k = 1, \ldots, p-1$ and $\ell = 0, 1, \ldots, p^n - 1$.

Hence for $m = kp^n + \ell$ with $k \in \{1, \ldots, p-1\}$ and $\ell \in \{0, 1, \ldots, p^n - 1\}$ we have

$$\frac{1}{m^\alpha} S_m(\mathbf{c}) = (1-\xi)^\alpha \frac{1}{k^\alpha} S_k(\mathbf{c}) + \xi^\alpha C_k(\mathbf{c}) \frac{1}{\ell^\alpha} S_\ell(\mathbf{c}) \tag{3.14}$$

where $\xi = \frac{\ell}{kp^n+\ell}$ with a certain $\ell \in \{0, 1, \ldots, p^n - 1\}$ so that $0 \leq \xi < \frac{1}{k+1}$. By (3.14) and (3.4) we get

$$\begin{aligned} K_{n+1} &\leq (1-\xi)^\alpha \left| \frac{1}{k^\alpha} S_k(\mathbf{c}) \right| + \xi^\alpha |C_k(\mathbf{c})| \left| \frac{1}{\ell^\alpha} S_\ell(\mathbf{c}) \right| \\ &\leq (1-\xi)^\alpha \left| \frac{1}{k^\alpha} S_k(\mathbf{c}) \right| + \xi^\alpha K_n \end{aligned}$$

where $k \in \{1, \ldots, p-1\}$, $\ell \leq p^n$ and in view of $K_n \leq K_{n+1}$ it follows

$$(1-\xi^\alpha) K_n \leq (1-\xi)^\alpha \left| \frac{1}{k^\alpha} S_k(\mathbf{c}) \right|.$$

Note $1 - \xi^\alpha > 0$ since $0 \leq \xi < \frac{1}{k+1}$ and $\alpha > 0$, cf. (3.8). Consequently,

$$K_n \leq \frac{(1-\xi)^\alpha}{1-\xi^\alpha} \left| \frac{1}{k^\alpha} S_k(\mathbf{c}) \right| \leq M_k \left| \frac{1}{k^\alpha} S_k(\mathbf{c}) \right| \qquad (k \in \{1, \ldots, p-1\}) \tag{3.15}$$

with

$$M_k = \max_{0 \le x \le \frac{1}{k+1}} \frac{(1-x)^\alpha}{1-x^\alpha}.$$

In view of (3.8) the function $f(x) = (1-x)^\alpha/(1-x^\alpha)$ is increasing so that we get $M_k = f(\frac{1}{k+1}) = \frac{k^\alpha}{(k+1)^\alpha - 1}$ and

$$K_n \le \frac{1}{(k+1)^\alpha - 1} |S_k(\mathbf{c})| \qquad (k \in \{1, \dots, p-1\}).$$

From (3.1) we get in view of $|C_j(\mathbf{c})| \le 1$ that $|S_k(\mathbf{c})| \le k$ so that

$$K_n \le \frac{k}{(k+1)^\alpha - 1} \qquad (k \in \{1, \dots, p-1\}).$$

The function $g(x) = \frac{x}{(x+1)^\alpha - 1}$ is increasing in $[1, p-1]$ so that $K_n \le g(p-1) = \frac{p-1}{p^\alpha - 1}$ which yields the assertion. □

**Remark 3.3** If we carry out the foregoing considerations with the coefficient $c_j^*$ of the reversed equation (2.8) instead of $c_j$ then in view of (2.9) and (3.3) we have $\mathbf{c}^* = \max\{|c_0^*|, \dots, |c_{p-1}^*|\} = \mathbf{c}$, and hence with the same $\alpha$ from (3.6) we find that the corresponding coefficients $C_j^*(\mathbf{c})$ satisfy $|C_j^*(\mathbf{c})| \le 1$ and that the sums $\frac{1}{k^\alpha} S_k^*(\mathbf{c})$ are bounded where

$$K^* := \sup_k \left| \frac{1}{k^\alpha} S_k^*(\mathbf{c}) \right| \tag{3.16}$$

can be estimates similarly as in (3.10). So

$$K^* \le \frac{p-1}{p^\alpha - 1}. \tag{3.17}$$

**Lemma 3.4** *If $|c_j| < 1$ for all $j \in \{0, 1, \dots, p-1\}$ then for $0 \le t \le 1$, $n \in \mathbb{N}$ and $k \in \{0, 1, \dots, p-1\}$ we have*

$$\left| \varphi\left(\frac{k+t}{p^n}\right) - \varphi\left(\frac{k}{p^n}\right) \right| \le K \left(\frac{t}{p^n}\right)^\alpha \tag{3.18}$$

*and for $k \in \{1, 2, \dots, p\}$*

$$\left| \varphi\left(\frac{k-t}{p^n}\right) - \varphi\left(\frac{k}{p^n}\right) \right| \le K^* \left(\frac{t}{p^n}\right)^\alpha. \tag{3.19}$$

**Proof:** We only prove (3.18). For $t = \frac{k}{p^n}$ with $0 \le k \le p^n$ the representation (2.13) with $c = \mathbf{c}$ implies

$$\frac{\varphi(t)}{t^\alpha} = \frac{\varphi(\frac{k}{p^n})}{(\frac{k}{p^n})^\alpha} = \frac{1}{k^\alpha} \sum_{j=0}^{k-1} C_j(\mathbf{c}) = \frac{1}{k^\alpha} S_k(\mathbf{c})$$

in view of (3.7). By Lemma 3.2 it follows

$$\frac{|\varphi(t)|}{t^\alpha} \le K$$

for these $t$ and hence also for arbitrary $t \in (0,1]$ by continuity. By (2.13) with $c = \mathbf{c}$ we have in view of (3.7)

$$\varphi\left(\frac{k+t}{p^n}\right) - \varphi\left(\frac{k}{p^n}\right) = \frac{1}{p^{\alpha n}} C_k(\mathbf{c})\varphi(t)$$

and using (3.4) we get

$$\left|\varphi\left(\frac{k+t}{p^n}\right) - \varphi\left(\frac{k}{p^n}\right)\right| \le \left(\frac{t}{p^n}\right)^\alpha \frac{|\varphi(t)|}{t^\alpha} \le \left(\frac{t}{p^n}\right)^\alpha K.$$

In the same way using (2.7) it follows (3.19). □

**Proposition 3.5** *If $|c_j| < 1$ for $j = 0, \ldots, p-1$ then for arbitrary $x, y \in [0,1]$ the solution $\varphi$ satisfies the inequality*

$$|\varphi(x) - \varphi(y)| \le \frac{p^{1-\alpha}(p-1)}{p^\alpha - 1}|x-y|^\alpha$$

*with $\alpha$ from (3.6).*

**Proof:** For given $x, y \in [0,1]$ with $h = y - x > 0$ we assume that

$$\frac{1}{p^n} \le h < \frac{1}{p^{n-1}}.$$

Let be $k = [p^n x]$ and $t_\mu = \frac{k+\mu}{p^n}$ $(\mu = 0, 1, \ldots)$. Then we have

$$t_0 \le x < t_1 < \ldots < t_m < y \le t_{m+1}$$

where $1 \le m \le p-1$ since $t_1 = \frac{k+1}{p^n} \le x + \frac{1}{p^n} \le x + h = y$ and $t_{p+1} = \frac{k+p+1}{p^n} > x + \frac{1}{p^{n-1}} > x + h = y$. We use

$$|\varphi(y) - \varphi(x)| \le |\varphi(t_1) - \varphi(x)| + |\varphi(y) - \varphi(t_m)| + \sum_{\mu=2}^{m-1} |\varphi(t_{\mu+1}) - \varphi(t_\mu)|.$$

We denote $a_1 = t_1 - x$, $a_k = t_k - t_{k-1}$ for $k = 2, \ldots, m-1$, and $a_m = y - t_m$ then $a_1 + \ldots + a_m = y - x$ and by Lemma 3.4

$$|\varphi(y) - \varphi(x)| \le K^* a_1^\alpha + K \sum_{\mu=2}^{m} a_\mu^\alpha \le K_{\max}\left(a_1^\alpha + \ldots + a_m^\alpha\right)$$

with $K_{\max} := \max\{K, K^*\}$. According to (3.8) the function $t \mapsto t^\alpha$ is concave and applying Jensen's inequality

$$\frac{a_1^\alpha + \ldots + a_m^\alpha}{m} \le \left(\frac{a_1 + \ldots + a_m}{m}\right)^\alpha$$

we find in view of $m \le p$ and (3.8)

$$|\varphi(y) - \varphi(x)| \le K_{\max} m^{1-\alpha}(y-x)^\alpha \le K_{\max} p^{1-\alpha}(y-x)^\alpha.$$

Finally, from (3.9) and (3.17) we get

$$K_{\max} p^{1-\alpha} \le p^{1-\alpha}\frac{p-1}{p^\alpha - 1}$$

and the proposition is proved. □

Now we know that $\varphi$ is Hölder continuous with exponent $\alpha$ from (3.6). Next we show that $\alpha$ is the optimal Hölder exponent and we determine also the optimal Hölder coefficient.

**Theorem 3.6** *If $|c_j| < 1$ for $j = 0, \ldots, p-1$ then the solution $\varphi$ of the equation (1.2) is Hölder continuous with the optimal Hölder exponent $\alpha$ from (3.6), i.e.*

$$\alpha = \min\left\{-\log_p |c_0|, \ldots, -\log_p |c_{p-1}|\right\}$$

*where $0 < \alpha < 1$, cf. (3.8), and the optimal Hölder coefficient*

$$A := \sup_{k,\ell} \frac{1}{k^\alpha}\left|\sum_{j=0}^{k-1} C_{\ell+j}(\mathbf{c})\right| \tag{3.20}$$

*which satisfies*

$$1 \le A \le \frac{p^{1-\alpha}(p-1)}{p^\alpha - 1}, \tag{3.21}$$

*i.e. we have*

$$|\varphi(x) - \varphi(y)| \le A\,|x-y|^\alpha \tag{3.22}$$

*for arbitrary $x, y \in [0,1]$.*

**Proof:** **1.** First we show (3.22) with $\alpha$ from (3.6) and $A$ from (3.20). For $y = \frac{\ell}{p^n}$ and $x = y + \frac{k}{p^n}$ with $0 \le \ell < k + \ell \le p^n$ the representation (2.13) with $c = \mathbf{c}$ implies

$$\frac{\varphi(x) - \varphi(y)}{(x-y)^\alpha} = \frac{\varphi(\frac{k+\ell}{p^n}) - \varphi(\frac{\ell}{p^n})}{(\frac{k}{p^n})^\alpha} = \frac{1}{k^\alpha}\sum_{j=\ell}^{k+\ell-1} C_j(\mathbf{c})$$

in view of (3.7). Hence, we get (3.22) for p-adic rational $x, y \in [0,1]$ where $A$ is finite by Proposition 3.5. Continuity of $\varphi$ implies that (3.22) is valid for all $x$ and $y$ in $[0,1]$.

**2.** We show that $\alpha$ is the optimal Hölder exponent. Assume that $\varphi$ is Hölder continuous with an exponent $\beta > \alpha$, i.e. for all $x, y \in [0,1]$ we have

$$|\varphi(x) - \varphi(y)| \le B|x-y|^\beta \tag{3.23}$$

with a certain constant $B$.

From (1.5) we get for $k = 0, 1, \dots, p-1$ by induction on $n$ that

$$\varphi\left(\frac{k(p^n-1)+t(p-1)}{p^n(p-1)}\right) = b_k\sum_{\nu=0}^{n-1} c_k^\nu + c_k^n\varphi(t) \qquad (0 \le t \le 1).$$

Putting $t = 0$ and $t = 1$ we get in view of $\varphi(0) = 0$ and $\varphi(1) = 1$ that

$$\varphi\left(\frac{k(p^n-1)+p-1}{p^n(p-1)}\right) - \varphi\left(\frac{k(p^n-1)}{p^n(p-1)}\right) = c_k^n. \tag{3.24}$$

Now we choose $k \in \{0, 1, \dots, p-1\}$ such that $|c_k| = \mathbf{c}$, cf. (3.3). In (3.24) we put $y = \frac{k(p^n-1)}{p^n(p-1)}$, $x = y + \frac{1}{p^n}$ and obtain in view of $x - y = \frac{1}{p^n}$, $|c_k| = \mathbf{c}$ and (3.6) that

$$|\varphi(x) - \varphi(y)| = (x-y)^\alpha.$$

According to (3.23) we get

$$\left(\frac{1}{p^n}\right)^\alpha \le B\left(\frac{1}{p^n}\right)^\beta,$$

i.e. $p^{n(\beta-\alpha)} \le B$, which yields a contradiction for large $n$. Hence, $\alpha$ is the optimal Hölder exponent and it follows that $A$ from (3.20) is the optimal Hölder coefficient. The estimate $A \ge 1$ follows from (3.22) with $x = 0$, $y = 1$ in view of $\varphi(0) = 0$, $\varphi(1) = 1$. The above estimate of $A$ follows from Proposition 3.5. □

**Remark 3.7** Note that in limit case $\alpha = 1$ we get $A = 1$ in accordance with $\varphi(x) = x$ for $0 \le x \le 1$.

A detail discussion of the Hölder continuity of de Rham's function and of solutions of certain two-scale difference equations you can find in [2, Section 2 and Section 5.2]. In [11, Proposition 10.1] it was shown the Hölder continuity of Cantor's function with optimal exponent $\alpha = \frac{\log 2}{\log 3}$ and coefficient $A = 1$.

# 4 Differentiability

As before we exclude the case $c_j = \frac{1}{p}$ for all $j \in \{0, 1, \dots, p-1\}$ where $\varphi(x) = x$ for $0 \le x \le 1$. First we give a general statement on the differentiability.

## 4.1 General statements

We start with the following simple lemma, cf. [15].

**Lemma 4.1** *Let* $f : [0,1] \mapsto \mathbb{R}$ *have a finite right-hand derivative* $f'_+(x_0)$ *at the point* $x_0 \in [0,1)$. *If* $(u_n)$ *and* $(v_n)$ *are sequences in* $[0,1]$ *such that* $x_0 < u_n < v_n$, $v_n \to x_0$ *and* $u_n - x_0 \leq L(v_n - u_n)$ *with a constant* $L$ *then*

$$\frac{f(v_n) - f(u_n)}{v_n - u_n} \to f'_+(x_0) \qquad (n \to \infty).$$

**Proposition 4.2** *If the solution* $\varphi$ *of* (1.2) *is differentiable at* $x_0$ *then* $\varphi'(x_0) = 0$.

**Proof:** Assume, at $x_0 \in [0,1)$ there exists the finite derivative $\varphi'(x_0) \neq 0$. For $n \in \mathbb{N}$ and $k = 0, 1, \ldots, p^n - 1$ we put $x_{k,n} = \frac{k}{p^n}$ and $N_{a,b} = \{k \in \mathbb{N} : a \leq k \leq b\}$. If $x_{k',n} \leq x_0 < x_{k'+1,n}$ then for each $k \in N_{k'+1,k'+2p-1}$ we put $u_{k,n} = x_{k,n}$ and $v_{k,n} = x_{k+1,n}$ so that $x_0 < u_{k,n} < v_{k,n}$ and $u_{k,n} - x_0 \leq p(v_{k,n} - u_{k,n})$. Applying (2.10) with $t = 1$ we get

$$\frac{\varphi(v_{k,n}) - \varphi(u_{k,n})}{v_{k,n} - u_{k,n}} = p^n c_0^n C_k$$

just as

$$\frac{\varphi(v_{k+1,n}) - \varphi(u_{k+1,n})}{v_{k+1,n} - u_{k+1,n}} = p^n c_0^n C_{k+1}.$$

In view of $\varphi'_+(x_0) \neq 0$ it follows by Lemma 4.1 that for $k \in N_{k'+1,k'+p}$ we have

$$\frac{C_{k+1}}{C_k} \to 1 \qquad (n \to \infty).$$

The set $N_{k'+1,k'+2p-1}$ contains a section of the form $N_{d,d+p-2}$ with $d = pk_0 < k' + p$. For $k \in N_{d,d+p-2}$, i.e. $k = pk_0 + j$ with $j = 0, 1, \ldots, p-2$ we have by (2.3) with $c = c_0$ that $C_k = C_{pk_0+j} = \frac{c_j}{c_0} C_{k_0}$ and it follows

$$\frac{c_{j+1}}{c_j} \to 1 \qquad (n \to \infty),$$

i.e. $c_{j+1} = c_j$. So by (1.3) it follows $c_j = \frac{1}{p}$ for $j = 0, \ldots, p-1$. $\square$

**Proposition 4.3** *The set* $E$ *of points* $x \in [0,1]$ *where* $\varphi$ *is differentiable has the Lebesgue measure* 0 *or* 1.

**Proof:** The set $E$ is Lebesgue measurable with the measure $|E|$. We show that $E$ is homogeneous, that means for each nonempty interval $[a,b]$ in $[0,1]$ we have $|E \cap [a,b]| = (b-a)|E|$. Equation (2.10) with $c = c_0$, $C_k = C_k(c_0)$ implies

$$\frac{1}{p^n} \varphi' \left( \frac{k+t}{p^n} \right) = c_0^n C_k \varphi'(t) \quad (t \in E).$$

Put $E_{k,n} := E \cap [\frac{k}{p^n}, \frac{k+1}{p^n}]$ we have $|E_{k,n}| = |E_{k',n}|$ $(0 \leq k, k' < p^n)$ and hence

$$E = \bigcup_{k=0}^{p^n-1} E_{k,n}$$

implies $|E_{k,n}| = \frac{1}{p^n}|E|$. It follows $|E \cap [a,b]| = (b-a)|E|$ for each interval $[a,b] \subset [0,1]$ and hence $|E| = 0$ or $|E| = 1$ by a theorem of Lebesgue. $\square$

### 4.2 Special difference quotients

Now, for given $x \in [0,1]$ we investigate the special difference quotients

$$\Delta_n(x) := \frac{\varphi(\frac{k+1}{p^n}) - \varphi(\frac{k}{p^n})}{1/p^n} \tag{4.1}$$

with $k = [p^n x]$, i.e.

$$\frac{k}{p^n} \le x < \frac{k+1}{p^n}. \tag{4.2}$$

Applying (2.10) with $c = c_0$ and $t = 1$ we get in view of $\varphi(1) = 1$

$$\Delta_n(x) = p^n c_0^n C_k. \tag{4.3}$$

In order to get a suitable representation for $C_k$ we need a mean value $M$. For $\lambda_j \in [0,1]$ with $\lambda_0 + \cdots + \lambda_{p-1} = 1$ we introduce the mean value $M = M(\lambda_0, \ldots, \lambda_{p-1})$ by

$$M := \prod_{j=0}^{p-1} |c_j|^{\lambda_j}. \tag{4.4}$$

**Lemma 4.4** *Let $x \in [0,1]$ and $n \in \mathbb{N}$. Then for $C_k = C_k(c_0)$ from (2.5) with $k = [p^n x]$ we have*

$$C_k = \frac{1}{c_0^n} \prod_{j=0}^{p-1} c_j^{s_j(k)} \tag{4.5}$$

*with*

$$s_0(k) + s_1(k) + \ldots + c_{p-1}(k) = n \tag{4.6}$$

*where $s_j(k)$ is the number of the digit $j$ in the $p$-adic representation of $k$. Further*

$$|C_k|^{1/n} = \frac{1}{|c_0|}\, e_n(x)\, M(\lambda_0, \ldots, \lambda_{p-1}) \tag{4.7}$$

*where*

$$e_n(x) := \prod_{j=0}^{p-1} |c_j|^{\varepsilon_j(n)} \tag{4.8}$$

*with $\varepsilon_j(n) = \frac{1}{n} s_j(k) - \lambda_j$ .*

**Proof:** If $x = 0, \xi_1 \xi_2 \ldots$ is the $p$-adic expansion of $x$ then $k = k(n) = [p^n x]$ has the form $k = \xi_n + \xi_{n-1} p + \ldots + \xi_1 p^{n-1}$. So $s_0(k) + \ldots + s_{p-1}(k) = n$ and by (2.5) we get

$$C_k = \prod_{j=0}^{p-1} \left(\frac{c_j}{c_0}\right)^{s_j(k)} = \frac{1}{c_0^n} \prod_{j=0}^{p-1} c_j^{s_j(k)},$$

i.e. (4.5) is proved. Formula (4.7) with (4.8) is a simple consequence of (4.5). $\square$

**Lemma 4.5** *Let $x \in [0,1]$ and $n \in \mathbb{N}$. Then for $\Delta_n(x)$ from (4.1) we have*

$$|\Delta_n(x)| = \prod_{j=0}^{p-1} a_j^{s_j(k)}$$

*with $a_j = p\,|c_j|$ so that $a_0 a_1 \dots a_{p-1} = 1$.*

**Proof:** Formulas (4.3) and (4.5) imply

$$\Delta_n(x) = p^n \prod_{j=0}^{p-1} c_j^{s_j(k)}.$$

In view of (4.6) it follows

$$\Delta_n(x) = \prod_{j=0}^{p-1} (p\,c_j)^{s_j(k)}$$

which proved the assertion. □

Next we consider special sets of real numbers, cf. [13, Chapter 10]. Let $x = 0, \xi_1\xi_2 \dots$ be the representation of a number $x \in (0,1)$ to the base $p$ and $d_j(x|_n)$ the total number of occurrence of the digit $j \in \{0, 1, \dots, p-1\}$ in the first $n$ places $0, \xi_1 \dots \xi_{n-1}$. That means

$$d_j(x|_n) = s_j(k) \tag{4.9}$$

where $k = [p^n x]$. For $\lambda_j \in [0,1]$ with $\lambda_0 + \dots + \lambda_{p-1} = 1$ let $F = F(\lambda_0, \dots, \lambda_{p-1})$ be the set

$$F := \left\{ x \in [0,1] : \lim_{n\to\infty} \frac{1}{n} d_j(x|_n) = \lambda_j \qquad \forall \quad j = 0, 1, \dots, p-1 \right\}. \tag{4.10}$$

It is known that $F$ has the Hausdorff dimension

$$dim_H F = -\frac{1}{log\,p} \sum_{j=0}^{p-1} \lambda_j \log \lambda_j \tag{4.11}$$

with the convention $0 \log 0 = 0$, cf. [13]. Further, the numbers $x \in F(p^{-1}, \dots, p^{-1})$ are called *normal numbers* with respect to the base $p$ and Borel's normal number theorem says that $F(p^{-1}, \dots, p^{-1})$ is a set of Lebesgue measure 1.

The following proposition is the basis for the investigation of $\varphi$ concerning the differentiability.

**Proposition 4.6** *For $x \in F(\lambda_0, \dots, \lambda_{p-1})$ we have*

$$\lim_{n\to\infty} |\Delta_n(x)|^{1/n} = p\,M(\lambda_0, \dots, \lambda_{p-1}) \tag{4.12}$$

*with $M$ from (4.4).*

**Proof:** By Lemma 4.4 and (4.3) we get

$$|\Delta_n(x)|^{1/n} = p\, e_n(x)\, M(\lambda_0, \dots, \lambda_{p-1}).$$

Using (4.9) for $x \in F(\lambda_0, \dots, \lambda_{p-1})$ we have $\frac{1}{n} s_j(k) = \frac{1}{n} d_j(x|_n) \to \lambda_j$ as $n \to \infty$. Hence $e_n(x) \to 1$ as $n \to \infty$ and it follows the assertion. □

### 4.3 The case $p\,M < 1$

We need further lemmata.

**Lemma 4.7** *Let be $x \in F(\lambda_0, \dots, \lambda_{p-1})$ with the $p$-adic expansion $x = 0, \xi_1 \xi_2 \dots$ where $\xi_{n-j} = p-1$ for $j = 1, 2, \dots, r_n$. If $\lambda_{p-1} < 1$ then $\frac{r_n}{n} \to 0$ as $n \to \infty$.*

**Proof:** For $x \in F$ we have $\frac{1}{n-r_n} d_{p-1}(x|_{n-r_n}) \to \lambda_{p-1}$ and $\frac{1}{n} d_{p-1}(x|_n) \to \lambda_{p-1}$ as $n \to \infty$. By supposition we have $d_{p-1}(x|_n) = d_{p-1}(x|_{n-r_n}) + r_n$ and hence

$$\frac{1}{n} d_{p-1}(x|_n) = \frac{n - r_n}{n} \frac{1}{n - r_n} d_{p-1}(x|_{n-r_n}) + \frac{r_n}{n}.$$

Certainly $0 \le \frac{r_n}{n} \le 1$, i.e. the sequence $\frac{r_n}{n}$ is bounded. If $s$ is the limit of a convergent subsequence then in view of (4.10) it follows $\lambda_{p-1} = (1-s)\lambda_{p-1} + s$, i.e $(1 - \lambda_{p-1})s = 0$ and hence $s = 0$ since $\lambda_{p-1} < 1$. So $\frac{r_n}{n} \to 0$ as $n \to \infty$. □

**Lemma 4.8** *Let be $x \in F(\lambda_0, \dots, \lambda_{p-1})$ with $\lambda_{p-1} < 1$ and $k = k(n) = [p^n x]$ then for $\mu = 0, 1, \dots, p$ we have*

$$\lim_{n \to \infty} \left| \frac{C_{k(n)+\mu}}{C_{k(n)}} \right|^{1/n} = 1$$

*with $C_k = C_k(c_0)$ from (2.5).*

**Proof:** Let be $x = 0, \xi_1 \xi_2 \dots$ the $p$-dic expansion of $x$ then $k = k(n) = [p^n x]$ has the form $k = \xi_n + \xi_{n-1} p + \dots + \xi_1 p^{n-1}$. For $\mu \ge 0$ with $\xi_{n-1} \le \xi_{n-1} + \mu < p$ we have $k + \mu = (\xi_n + \mu) + \xi_{n-1} p + \dots + \xi_1 p^{n-1}$ and by (2.5) it holds

$$C_{k+\mu} = C_k \frac{C_{\xi_n + \mu}}{C_{\xi_n}}.$$

Now we consider $\mu \in \{1, \dots, p\}$ with $p \le \xi_{n-1} + \mu \le 2p - 1$. If $\xi_{n-1} < p - 1$ then we have $k + \mu = (\xi_n + \mu - p) + (\xi_{n-1} + 1)p + \xi_{n-2} p^2 + \dots + \xi_1 p^{n-1}$ and by (2.5) it holds

$$C_{k+\mu} = C_k \frac{C_{\xi_n + \mu - p}}{C_{\xi_n}} \frac{C_{\xi_{n-1}+1}}{C_{\xi_{n-1}}}.$$

If $\xi_{n-j} = p-1$ for $j = 1, \ldots, n_r$ and $\xi_{n-n_r-1} < p-1$ then $k$ has the representation

$$k = \xi_n + (p-1)p + \ldots + (p-1)p^{r_n} + \xi_{n-n_r-1}p^{r_n+1} + \ldots + \xi_1 p^{n-1}$$

and we have

$$k + \mu = (\xi_n + \mu - p) + (\xi_{n-r_n-1} + 1)p^{r_n+1} + \ldots + \xi_1 p^{n-1}\,.$$

According to (2.5) we get

$$C_{k+\mu} = C_k \frac{C_{\xi_n+\mu-p}}{C_{\xi_n}} \left(\frac{C_{p-1}}{C_0}\right)^{r_n} \frac{C_{\xi_{n-r_n-1}+1}}{C_{\xi_{n-r_n}}}.$$

Put $C = \max\{|C_i|/|C_j|\}$ $(i,j = 0,1,\ldots,p-1)$ then we get

$$\frac{1}{C^{r_n+2}} \leq \left|\frac{C_{k+\mu}}{C_k}\right| \leq C^{r_n+2}$$

and in view of Lemma 4.7 it follows the assertion. □

**Proposition 4.9** *If $p\,M(\lambda_0,\ldots,\lambda_{p-1}) < 1$ where $\lambda_{p-1} < 1$ then the solution $\varphi$ of (1.2) is differentiable at each point $x \in F(\lambda_0,\ldots,\lambda_{p-1})$ with $\varphi'(x) = 0$.*

**Proof:** Choose $\varepsilon > 0$ so that

$$q := (1+\varepsilon)^3\, p\, M(\lambda_0,\ldots,\lambda_{p-1}) < 1.$$

For fixed $x \in F(\lambda_0,\ldots,\lambda_{p-1})$ it holds $e_n(x) \to 1$ as $n \to \infty$, cf. (4.8), (4.10) and (4.9). Hence there is a number $n_0$ such that for $n \geq n_0$ we have

$$e_n(x) < 1 + \varepsilon, \tag{4.13}$$

$$K^{1/n} < 1 + \varepsilon \tag{4.14}$$

with $K = \max|\varphi(t)|$ for $0 \leq t \leq 1$ and by Lemma 4.8

$$|C_{k+\mu}|^{1/n} < (1+\varepsilon)|C_k|^{1/n} \tag{4.15}$$

where $k = [p^n x]$. Now, let $y = x + h < 1$ with $h > 0$ (the case $h < 0$ is analogous) and

$$\frac{1}{p^n} \leq h < \frac{1}{p^{n-1}}$$

with $n \geq n_0$. Note that $h \to 0$ is equivalent to $n \to \infty$. Put $t_\mu = \frac{k+\mu}{p^n}$ $(\mu = 0,1,\ldots)$ then we have $t_0 < x < t_1 < \ldots < t_m < x + h \leq t_{m+1}$ where $1 \leq m \leq p-1$ since $t_1 = \frac{k+1}{p^n} \leq x + \frac{1}{p^n} \leq x + h$ and $t_{p+1} = \frac{k+p+1}{p^n} > x + \frac{1}{p^{n-1}} > x + h$. We use

$$|\varphi(x+h) - \varphi(x)| \leq |\varphi(x+h) - \varphi(t_m)| + |\varphi(t_1) - \varphi(x)| + \sum_{\mu=1}^{m-1} |\varphi(t_{\mu+1}) - \varphi(t_\mu)|.$$

Put $x = \frac{k+1-t}{p^n}$ with suitable $0 \le t < 1$ then by (2.10) (with $a = c_0$, $C_k = C_k(c_0)$) we have

$$\varphi(x) - \varphi(t_1) = c_0^n C_{k+1}\varphi(1-t)$$

and hence

$$\frac{|\varphi(t_1) - \varphi(x)|}{h} = \frac{1}{h}|c_0|^n|C_{k+1}||\varphi(1-t)| \le p^n|c_0|^n|C_{k+1}|K$$

where $K = \max|\varphi|$. Applying Lemma 4.4 we get

$$p|c_0||C_{k+1}|^{1/n}K^{1/n} = p|c_0||C_k|^{1/n}\left|\frac{C_{k+1}}{C_k}\right|^{1/n}K^{1/n} = pMe_n(x)\left|\frac{C_{k+1}}{C_k}\right|^{1/n}K^{1/n}$$

and using (4.13), (4.14) and (4.15) it follows

$$p|c_0||C_{k+1}|^{1/n}K^{1/n} < (1+\varepsilon)^3 pM = q$$

so that

$$\frac{|\varphi(x) - \varphi(t_1)|}{h} < q^n. \tag{4.16}$$

Since $t_m < x + h \le t_{m+1}$ we have $x + h = \frac{k+m+\tau}{p^n}$ with suitable $0 < \tau \le 1$ and by (2.10)

$$\varphi(x+h) - \varphi(t_m) = c_0^n C_{k+m}\varphi(\tau).$$

Therefore

$$\frac{|\varphi(x+h) - \varphi(t_m)|}{h} \le \frac{1}{h}|c_0|^n|C_{k+m}|K \le p^n|c_0|^n|C_{k+m}|K < q^n \tag{4.17}$$

where we have again used (4.13), (4.14) and (4.15).

Moreover, by (2.10) it holds

$$\varphi(t_{\mu+1}) - \varphi(t_\mu) = c_0 C_{k+\mu}$$

and hence again

$$\frac{|\varphi(t_{\mu+1}) - \varphi(t_\mu)|}{h} = \frac{1}{h}|c_0|^n|C_{k+\mu}| \le p^n|c_0|^n|C_{k+\mu}| < q^n. \tag{4.18}$$

Form (4.16), (4.17) and (4.18) it follows in view of $m \le p-1$

$$\frac{|\varphi(x+h) - \varphi(x)|}{h} < (p+1)q^n.$$

This implies $\varphi'_+(x) = 0$. In the same way $\varphi'_-(x) = 0$. $\square$

### 4.4 The case $pM = 1$

We investigate $\Delta_n(x)$ from (4.1) under the condition

$$p\,|c_0c_1\cdots c_{p-1}|^{1/p} = 1. \tag{4.19}$$

The following proof due to A. Meister (personal communication).

**Lemma 4.10** *Assume that it holds (4.19) and that $a_j = p|c_j|$ for $j = 0, 1, \ldots, p-1$. If not $a_0 = a_1 = \ldots = a_{p-1} = 1$ then the set of $x$ with the property $\Delta_n(x) \to 0$ as $n \to \infty$ has the measure zero.*

**Proof:** Let $x = 0, \xi_1\xi_2\ldots$ where the digits $\xi_j$ are independent and identically distributed on the discrete set $\{0, 1, \ldots, p-1\}$. Since

$$d_j(x|_n) = \sum_{k=1}^{n} \chi_j(\xi_k)$$

we have by Lemma 4.5 and (4.9)

$$\log|\Delta_n(x)| = \sum_{k=1}^{n}\sum_{j=0}^{p-1} \chi_j(\xi_k)\log a_j = \sum_{k=1}^{n} \log a_{\xi_k}$$

where $\log a_{\xi_k}$ are independent and identically distributed. Moreover,

$$E(\log a_{\xi_k}) = \sum_{j=0}^{p-1} \frac{1}{p}\log a_j = \frac{1}{p}\log\left(\prod_{j=0}^{p-1} a_j\right) = 0$$

since by (4.19) we have $a_0a_1\cdots a_{p-1} = 1$, and it is

$$\sigma^2 = E(\log^2 a_{\xi_k}) = \sum_{j=0}^{p-1} \frac{1}{p}(\log^2 a_j) > 0$$

since not all $a_j$ are equal to 1. The law of iterated logarithm says

$$\limsup_{n\to\infty} \frac{\log|\Delta_n(x)|}{\sqrt{2\sigma^2 n\log\log n}} = +1 \qquad (a.s.)$$

and

$$\liminf_{n\to\infty} \frac{\log|\Delta_n(x)|}{\sqrt{2\sigma^2 n\log\log n}} = -1 \qquad (a.s.).$$

This implies the assertion. $\square$

### 4.5 On the differentiability of the solution

After the foregoing preparations we are able to give the main results concerning differentiability of the solution $\varphi$ of (1.2). As already mentioned in the Introduction we again exclude the trivial case $c_j = \frac{1}{p}$ for all $j = 0, 1, \ldots, p-1$.

**Theorem 4.11** *The solution $\varphi$ of (1.2) has the property:*

**1.** *If* $\min |c_j| \geq \frac{1}{p}$ *then $\varphi$ is nowhere differentiable in* $[0, 1]$.

**2.** *If* $\min |c_j| < \frac{1}{p}$ *then both sets, where $\varphi$ is differentiable and where $\varphi$ does not have a finite derivative, have positive Hausdorff dimension.*

**Proof:** **1.** If $|c_j| \geq \frac{1}{p}$ for all $j = 0, 1, \ldots, p-1$ then $a_j = p\,|c_j| \geq 1$ and for each $x \in [0, 1]$ we have by Lemma 4.5 that $|\Delta_n(x)| \geq 1$ for all $n \in \mathbb{N}$. So $\varphi$ is not differentiable at $x$ according to Proposition 4.2.

**2.** If $\min |c_j| < \frac{1}{p}$ then in view of (1.3) there are indices $k$ and $\ell$ such that $|c_k| < \frac{1}{p}$ and $|c_\ell| > \frac{1}{p}$. For the mean value (4.4) we have $M(\lambda_0, \ldots, \lambda_{p-1}) = |c_k| < \frac{1}{p}$ if $\lambda_k = 1$ and $\lambda_j = 0$ for $j \neq k$. Hence, there exist such $\lambda'_j > 0$ (with $\lambda'_k$ nearly by 1 and $\lambda'_{p-1} < 1$) that $p\,M(\lambda'_0, \ldots, \lambda'_{p-1}) < 1$. By Proposition 4.9 we have $\varphi'(x) = 0$ for $x \in F(\lambda'_0, \ldots, \lambda'_{p-1})$ and by (4.11) this set $F$ has positive Hausdorff dimension. Moreover, $M(\lambda_0, \ldots, \lambda_{p-1}) = |c_\ell| > \frac{1}{p}$ if $\lambda_\ell = 1$ and $\lambda_j = 0$ for $j \neq \ell$, so that there are $\lambda''_j > 0$ such that $p\,M(\lambda''_0, \ldots, \lambda''_{p-1}) > 1$. For $x \in F(\lambda''_0, \ldots, \lambda''_{p-1})$ it fails $\Delta_n(x) \to 0$ by Proposition 4.6 so that $\varphi$ is not differentiable at $x$ according to Proposition 4.2, and by (4.11) also this set $F$ has positive Hausdorff dimension. □

**Theorem 4.12** *The solution $\varphi$ of (1.2) has in $[0, 1]$ the property:*

**1.** *If* $p\,|c_0 c_1 \cdots c_{p-1}|^{1/p} < 1$ *then* $\varphi'(x) = 0$ *almost everywhere.*

**2.** *If* $p\,|c_0 c_1 \cdots c_{p-1}|^{1/p} \geq 1$ *then $\varphi$ is almost nowhere differentiable.*

**Proof:** We consider $x \in F(p^{-1}, \ldots, p^{-1})$ and remember that this set has the Lebesgue measure 1 by Borel's normal number theorem.

**1.** If $p\,|c_0 c_1 \cdots c_{p-1}|^{1/p} < 1$ then by Proposition 4.9 we have $\varphi'(x) = 0$ for each $x \in F(p^{-1}, \ldots, p^{-1})$.

**2.1.** If $p\,|c_0 c_1 \cdots c_{p-1}|^{1/p} = 1$ then by Proposition 4.10 the set of all $x \in F(p^{-1}, \ldots, p^{-1})$ with $\limsup |\Delta_n(x)| > 0$ has the measure 1. For all these $x$ the derivative does not exist according to Proposition 4.2.

**2.2.** If $p\,|c_0 c_1 \cdots c_{p-1}|^{1/p} > 1$ then for each $x \in F(p^{-1}, \ldots, p^{-1})$ we have according to Proposition 4.6 that $|\Delta_n(x)| \to \infty$ as $n \to \infty$ and hence the derivative does not exist owing to Proposition 4.2. □

**Remark 4.13** **1.** Note that Proposition 4.3 is a consequence of Theorem 4.12.

**2.** Assume that $\varphi$ is an increasing solution of (1.2) but not $\varphi(x) = x$ for all $x \in [0,1]$. Then by Proposition 2.5 together with Proposition 2.6 we have $c_j \geq 0$ for all $j = 0, 1, \ldots, p-1$ but not $c_j = \frac{1}{p}$ for all $j$ and in view of (1.3)

$$(c_0 c_1 \cdots c_{p-1})^{1/p} < \frac{c_0 + c_1 + \ldots + c_{p-1}}{p} = \frac{1}{p}$$

so that $\varphi'(x) = 0$ almost everywhere by Theorem 4.12. So for an increasing solution $\varphi$ of (1.2) we have besides of $\varphi(x) = x$ for $x \in [0,1]$ that $\varphi'(x) = 0$ almost everywhere.

## 5 Singular solutions

A nonconstant $\varphi : [0,1] \mapsto [0,1]$ is called (strictly) singular, if it is continuous and (strictly) increasing with $\varphi'(x) = 0$ almost everywhere. We remember that in case $c_j = \frac{1}{p}$ for $j \in \{0, 1, \ldots, p-1\}$ the solution $\varphi$ of (1.2) reads $\varphi(x) = x$ for $0 \leq x \leq 1$ and that we exclude this trivial case. As already mentioned in Remark 3.3.1 we use the parameter $c = c_0$ and write short $C_k$ for $C_k(c_0)$.

From Proposition 2.5 and Proposition 4.2 we get

**Proposition 5.1** *If $c_j \geq 0$ for all $j = 0, 1, \ldots, p-1$ then the solution $\varphi$ of (1.2) is a singular function and if $c_j > 0$ for all $j$ then it is strictly singular.*

**Lemma 5.2** *If $\varphi$ is a solution of equation (1.2) satisfying (1.4) then $\varphi$ cannot vanish in a neighborhood of $x = 0$.*

**Proof:** Assume that $\varphi(x) = 0$ for $x < \varepsilon_0$ where $\varepsilon_0 > 0$. In view of

$$\varphi\left(\frac{x}{p}\right) = c_0 \varphi(x) \qquad (0 \leq x \leq 1)$$

and $c_0 \neq 0$ implies $\varphi(x) = 0$ for $x < p\,\varepsilon_0$. In view of $p > 1$ it follows $\varepsilon_0 = 0$ since $\varphi(x) = 1$ for $x > 1$. □

**Proposition 5.3** *Let be $0 \leq c_j < 1$ with $\min c_j = 0$. Then the solution $\varphi$ of (1.2) is constant on the components $(a_i, b_i)$ of an open set $G$ with Lebesgue measure $|G| = 1$. The endpoints $a_i$ and $b_i$ are of the form $\frac{k}{p^n}$ where we have:*

- $\frac{k}{p^n} = a_i \iff C_{k-1} \neq 0, \quad C_k = 0,$
- $\frac{k}{p^n} = b_i \iff C_{k-1} = 0, \quad C_k \neq 0,$
- $\frac{k}{p^n} \in G \iff C_{k-1} = 0, \quad C_k = 0.$

**Proof:** Assume that $c_{k_0} = 0$ where $1 \le k_0 \le p-2$. From (1.5) it follows that $\varphi$ is constant on the interval

$$I_{k_0} = \left( \frac{k_0}{p}, \frac{k_0+1}{p} \right).$$

By repeated application of (1.5) we see that $\varphi$ is constant on the intervals

$$I_{k_1,k_0} = \left( \frac{k_1}{p} + \frac{k_0}{p^2}, \frac{k_1}{p} + \frac{k_0+1}{p^2} \right)$$

where $k_1 \neq k_0$, $0 \le k_1 \le p-1$, and in general

$$I_{k_{n-1},\ldots,k_0} = \left( \frac{k_{n-1}}{p} + \ldots + \frac{k_1}{p^{n-1}} + \frac{k_0}{p^n}, \frac{k_{n-1}}{p} + \ldots + \frac{k_1}{p^{n-1}} + \frac{k_0+1}{p^n} \right),$$

where $k_\nu \neq k_0$ and $0 \le k_\nu \le p-1$ for $\nu > 0$. Obviously, $I_{k_{n-1},\ldots,k_0}$ has the Lebesgue measure $|I_{k_{n-1},\ldots,k_0}| = \frac{1}{p^n}$. These intervals are pairwise different and hence the union $G_0$ has the Lebesgue measure

$$|G_0| = \sum_{n=1}^{\infty} \frac{(p-1)^{n-1}}{p^n} = \frac{1}{p(1-\frac{p-1}{p})} = 1.$$

The left endpoint of $I_{k_{n-1},\ldots,k_0}$ has the form $\frac{k}{p^n}$ with

$$k = k_{n-1}p^{n-1} + k_{n-2}p^{n-2} + \ldots + k_1 p + k_0$$

so that $c_{k_0} = 0$ implies $C_k = 0$, cf. (2.3). It follows from (2.10) that

$$\varphi\left(\frac{k+t}{p^n}\right) = \varphi\left(\frac{k}{p^n}\right) \qquad (0 \le t \le 1),$$

i.e. $\varphi$ is constant on $I_{k_{n-1},\ldots,k_0}$. If $G$ is an open set such that $\varphi$ is constant on each component $(a_i, b_i)$ of $G$ then $G_0 \subseteq G \subseteq [0,1]$ and hence $|G| = 1$ too.

Now let $(a_i, b_i)$ be a maximal interval where $\varphi$ is constant. Choose $n$ so large that $b_i - a_i > \frac{2}{p^n}$ then there is an integer $k$ such that

$$\frac{k-1}{p^n} < a_i \le \frac{k}{p^n} \tag{5.1}$$

and $\frac{k+1}{p^n} < b_i$, i.e. $\varphi$ is constant on $[\frac{k}{p^n}, \frac{k+1}{p^n}]$.

We show that $a_i = \frac{k}{p^n}$ and that $C_{k-1} \neq 0$, $C_k = 0$. By Lemma 5.2 $\varphi(x)$ cannot vanish in a neighborhood of $x = 0$ which is true also for $\varphi^*(x) = 1 - \varphi(1-x)$ since $c_{p-1} > 0$. Therefore in view of (5.1) equation (2.12) implies that $\varphi$ is not constant in a neighborhood of $\frac{k}{p^n}$ which implies $a_i = \frac{k}{p^n}$ and $C_{k-1} \neq 0$. Moreover, equation (2.10) for $t = 1$ yields

$$\varphi\left(\frac{k+1}{p^n}\right) = \varphi\left(\frac{k}{p^n}\right) + c_0^n C_k$$

and hence $c_0^n C_k = 0$ must be. It follows $C_k = 0$ since $c_0 > 0$.

Conversely, let be $C_k = 0$ and $C_{k-1} \neq 0$. Then equation (2.10) implies

$$\varphi\left(\frac{k+t}{p^n}\right) = \varphi\left(\frac{k}{p^n}\right) \qquad (0 \le t \le 1),$$

i.e. $\varphi$ is constant on $[\frac{k}{p^n}, \frac{k+1}{p^n}]$. Moreover, equation (2.12) implies that $\varphi$ is not constant in a neighborhood of $\frac{k}{p^n}$ so that it is a left endpoint $a_i$ of an interval of constancy. In the same manner the another assertions can be proved. □

In case $0 \le c_j < 1$ and $\min c_j = 0$ equation (1.2) can be written in the form

$$\varphi\left(\frac{x}{p}\right) = \sum_{n=0}^{q-1} c_{\gamma_n} \varphi(x - \gamma_n) \qquad (x \in \mathbb{R}) \tag{5.2}$$

where $q$ is an integer with $1 \le q \le p-1$ and where $\gamma_n$ are nonnegative integers with $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_{q-1} = p-1$. The characteristic polynomial of equation (5.2) reads $P(z) = c_0 + c_{\gamma_1} z^{\gamma_1} + \cdots + c_{p-1} z^{p-1}$ and (2.6) has the form

$$G(z) = \prod_{j=0}^{\infty} \frac{1}{c_0} P(z^{p^j}) = \sum_{n=0}^{\infty} C_{\gamma_n} z^{\gamma_n} \tag{5.3}$$

with strictly increasing integers $\gamma_n$ where it holds with $\varepsilon_\mu \in \{0, 1, \ldots, q-1\}$:

$$n = \sum_{\mu=0}^{m-1} \varepsilon_\mu q^\mu \qquad \Longrightarrow \qquad \gamma_n = \sum_{\mu=0}^{m-1} \gamma_{\varepsilon_\mu} p^\mu. \tag{5.4}$$

In particular, if $n = \sum_{\mu=0}^{m-1} (q-1) q^\mu = q^m - 1$ then $\gamma_n = \sum_{\mu=0}^{m-1} (p-1) p^\mu = p^m - 1$ and

$$\gamma_{qn+r} = p\gamma_n + \gamma_r \qquad (r \in \{0, 1, \ldots, q-1\}). \tag{5.5}$$

**Theorem 5.4** *The open intervals $J_{m,n} \subseteq [0,1]$ where the solution $\varphi$ of (5.2) is constant have the form*

$$J_{m,n} = \left(\frac{\gamma_{m-1}+1}{p^n}, \frac{\gamma_m}{p^n}\right) \qquad (n = 1, 2, \ldots, \quad m = 1, 2, \ldots, q^n - 1) \tag{5.6}$$

*provided that $\gamma_{m-1} + 1 < \gamma_m$.*

**Proof:** We apply Proposition 5.3. If $(a_i, b_i)$ is a maximal interval of constancy then by Proposition 5.3 and the definition of $\gamma_n$ we have $a_i = \frac{\gamma_k + 1}{p^n}$ and $b_i = \frac{\gamma_m}{p^n}$ with suitable $k$, $m$. Since the sequence $\gamma_n$ is strictly increasing it follows $k = m-1$, i.e. $(a_i, b_i) = J_{m,n}$ from (5.11) with the given indices there. □

**Remark 5.5** **1.** Observe that $J_{qm,n+1} = J_{m,n}$ since in view of (5.5) we have for the left endpoint

$$\gamma_{qm-1} + 1 = \gamma_{q(m-1)+q-1} + 1 = p\gamma_{m-1} + \gamma_{q-1} + 1 = p(\gamma_{m-1} + 1)$$

where we have used $\gamma_{q-1} = p - 1$, and for the right endpoint $\gamma_{qm} = p\gamma_m$. So we can see again that the nonempty intervals $J_{m,n}$ coincide or they are disjoint.

**2.** Note that

$$\sum_{m=1}^{q^n-1} |J_{m,n}| = \sum_{m=1}^{q^n-1} \frac{\gamma_m - \gamma_{m-1} - 1}{p^n} = \frac{\gamma_{q^n-1} - \gamma_0 - q^n}{p^n} = \frac{p^n - 1 - q^n}{p^n} \to 1$$

as $n \to \infty$.

**Example 5.6** (*Cantor's function.*) We know that Cantor's function $\varphi$ is the to [0,1] restricted solution of (1.2) with $c_0 = \frac{1}{2}$, $c_1 = 0$, $c_1 = \frac{1}{2}$, i.e.

$$\varphi\left(\frac{x}{3}\right) = \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(x-2) \qquad (x \in \mathbb{R})$$

satisfying (1.4). Here $P(z) = (1 + z^2)/2$ and the generating function (2.6) reads

$$G(z) = \prod_{j=0}^{\infty} \left(1 + z^{2 \cdot 3^j}\right) = \sum_{k=0}^{\infty} C_k z^k \tag{5.7}$$

where $C_k = 0$ if the triadic representation of $k$ contains the digit 1, elsewhere $C_k = 1$. Hence, $G$ can be written as

$$G(z) = \sum_{n=0}^{\infty} z^{\gamma_n} = 1 + z^2 + z^6 + z^8 + z^{18} + z^{20} + z^{24} + z^{26} + \dots \tag{5.8}$$

with strictly increasing exponents $\gamma_0 = 0$, $\gamma_1 = 2$, $\gamma_2 = 6$, $\gamma_3 = 8$ and so on. It holds with $\varepsilon_\mu \in \{0, 1\}$:

$$n = \sum_{\mu=0}^{m-1} \varepsilon_\mu 2^\mu \qquad \Longrightarrow \qquad \gamma_n = 2\sum_{\mu=0}^{m-1} \varepsilon_\mu 3^\mu \tag{5.9}$$

and it is easy to see that

$$\gamma_{k-1} + \gamma_{2^n-k} = 3^n - 1 \qquad (n = 1, 2, \dots, \quad k = 1, 2, \dots, 2^n). \tag{5.10}$$

The open intervals $J_{m,n}$ where Cantors function $\varphi$ is constant have the form

$$J_{m,n} = \left(\frac{\gamma_{m-1} + 1}{3^n}, \frac{\gamma_m}{3^n}\right) \qquad (n = 1, 2, \dots, \quad m = 1, 2, \dots, 2^n - 1) \tag{5.11}$$

with $\varphi(x) = \frac{m}{2^n}$ for $x \in J_{m,n}$.

## 6 Subadditivity

In this section we investigate the subadditivity of the solution $\varphi$ of (1.2), i.e.

$$\varphi(x+y) \leq \varphi(x)+\varphi(y) \tag{6.1}$$

for all $x, y \in \mathbb{R}$. For this purpose again we consider the sequence $S_k(\mathbf{c})$ from (3.1) with $c=\mathbf{c}$ from (3.3).

**Lemma 6.1** *Assume that $c_j \geq 0$ for all $j=0,1,\ldots,p-1$ and that for $0 \leq k, \ell < p$ we have*

$$S_k(\mathbf{c})+S_\ell(\mathbf{c}) \geq \begin{cases} S_{k+\ell}(\mathbf{c}) & \text{if} \quad k+\ell<p \\ S_{k+\ell-p}(\mathbf{c})+S_p(\mathbf{c}) & \text{if} \quad k+\ell \geq p \end{cases} \tag{6.2}$$

*then for all nonnegative integers $k, \ell$ it holds*

$$S_{k+\ell}(\mathbf{c}) \leq S_k(\mathbf{c})+S_\ell(\mathbf{c}). \tag{6.3}$$

**Proof:** First note that by (2.5) we have $C_j(\mathbf{c}) \geq 0$ for all $j \in \mathbb{N}$. We shall prove the inequality (6.3) for nonnegative integers $k, \ell < p^n$ by induction with respect to $n$ where as abbreviation we write $S_k$ in place of $S_k(\mathbf{c})$. For $n=0$ the inequality is true by (6.2). Assume that (6.3) is true for $0 \leq k, \ell < p^n$. For integers $0 \leq k, \ell < p^{n+1}$ we write $k=pk'+i$ and $\ell=p\ell'+j$ with $0 \leq k', \ell' < p^n$ and $i, j \in \{0,1,\ldots,p-1\}$. We consider two cases:

**1.** Let be $i+j<p$. Then in view of Lemma 3.1/(iii) we have

$$\begin{aligned} S_{p(k'+\ell')+i+j} &= S_p S_{k'+\ell'} + C_{k'+\ell'}(\mathbf{c}) S_{i+j} \\ &\leq S_p(S_{k'}+S_{\ell'}) + C_{k'+\ell'}(\mathbf{c})(S_i+S_j) \\ &\leq S_p S_{k'} + S_p S_{\ell'} + C_{k'}(\mathbf{c}) S_i + C_{\ell'}(\mathbf{c}) S_j \\ &= S_{pk'+i} + S_{p\ell'+j} \end{aligned}$$

where we have used that (3.4) and that $C_{k'+\ell'}(\mathbf{c}) \leq \min\{C_{k'}(\mathbf{c}), C_{\ell'}(\mathbf{c})\}$ according to (2.5). So $S_{k+\ell} \leq S_k + S_\ell$.

**2.** In case $i+j \geq p$ we have $0 \leq i+j-p < p-1$. Applying Lemma 3.1/(iii) and assumption of induction we get

$$S_{k'+\ell'+1} = S_{k'+\ell'} + C_{k'+\ell'}(\mathbf{c}) \leq S_{k'} + S_{\ell'} + C_{k'+\ell'}(\mathbf{c})$$

and

$$\begin{aligned} S_{p(k'+\ell'+1)+i+j-p} &= S_p S_{k'+\ell'+1} + C_{k'+\ell'+1}(\mathbf{c}) S_{i+j-p} \\ &\leq S_p\{S_{k'}+S_{\ell'}+C_{k'+\ell'}(\mathbf{c})\} + C_{k'+\ell'+1}(\mathbf{c}) S_{i+j-p} \\ &\leq S_p S_{k'} + S_p S_{\ell'} + C_{k'+\ell'}(\mathbf{c})(S_p + S_{i+j-p}) \\ &\leq S_p S_{k'} + S_p S_{\ell'} + C_{k'+\ell'}(\mathbf{c})(S_i + S_j) \end{aligned}$$

where we have used (6.2) and $C_{k'+\ell'+1}(\mathbf{c}) \le C_{k'+\ell'}(\mathbf{c})$ according to (2.5) and (3.4). Hence we have $S_{k+\ell} \le S_k + S_\ell$ again. □

**Theorem 6.2** *If (6.2) is satisfies then the solution $\varphi$ of (1.2) is subadditive, i.e.*

$$\varphi(x+y) \le \varphi(x) + \varphi(y) \qquad (x, y \in \mathbb{R}). \tag{6.4}$$

**Proof:** For $x = \frac{k}{p^n}$, $y = \frac{\ell}{p^n}$ in [0,1] with $x + y \le 1$ the assertion follows from (3.2) in view of (6.3), and for arbitrary $x, y \in [0,1]$ with $x + y \le 1$ by continuity of $\varphi$. Now from (1.4) it is easy to see that the inequality is true for all $x, y \in \mathbb{R}$. □

**Example 6.3** (*De Rham's function*) We know that de Rham's function $\varphi$ is the to [0,1] restricted solution $\varphi$ of (1.2) with $c_0 = a$, $c_1 = 1 - a$, $a \in (0,1)$, i.e.

$$\varphi\left(\frac{x}{2}\right) = a\varphi(x) + (1-a)\varphi(x-1) \qquad (x \in \mathbb{R})$$

satisfying (1.4), cf. Example 2.4. For $0 < a < \frac{1}{2}$ de Rham's function $\varphi$ fails to be subadditive since $2\varphi(\frac{1}{2}) = 2a < 1 = \varphi(1)$. In case $\frac{1}{2} \le a < 1$ we have $\mathbf{c} = \max\{a, 1-a\} = a$ and $C_k = C_k(a) = q^{s(k)}$ with $q = \frac{1-a}{a}$, where $s(k)$ denotes the number of ones in the dyadic representation of $k$, i.e. $C_0 = 1$, $C_1 = q$, $C_2 = q$, $C_3 = q^2$, $C_4 = q$, $C_5 = q^2$ and for $S_k = S_k(a)$ we have $S_1 = 1$, $S_2 = 1 + q$, $S_3 = 1 + 2q$, $S_4 = 1 + 2q + q^2$, $S_5 = 1 + 3q + q^2$. So inequality (6.2) is satisfies if $0 < q \le 1$, i.e. $\frac{1}{2} \le a < 1$ and for these $a$ we have (6.3), cf. [2, Lemma 2.2]. Hence, for $\frac{1}{2} \le a < 1$ the extended de Rham function is subadditive owing to Theorem 6.2.

Finally we consider once more two-scale difference equation (1.13).

**Example 6.4** (*Equation* (1.13)) For $0 < a < 1$ let $\varphi$ be the continuous solution of

$$\varphi\left(\frac{x}{3}\right) = a\varphi(x) + (1-2a)\varphi(x-1) + a\varphi(x-2) \qquad (x \in \mathbb{R})$$

satisfying (1.4). For $0 < a \le \frac{1}{2}$ the coefficients are nonnegative. In case $0 < a < \frac{1}{3}$ the solution $\varphi$ fails to be subadditive since $\varphi(\frac{2}{3}) = 1 - a > 2a = 2\varphi(\frac{1}{3})$. In case $\frac{1}{3} \le a \le \frac{1}{2}$ we have $\mathbf{c} = \max\{a, 1-2a, a\} = a$ and $C_k = C_k(a) = \varrho^{s_1(k)}$ with $\varrho = \frac{1-2a}{a}$, where $s_1(k)$ denotes the number of ones in the triadic representation of $k$. So $C_0 = 1$, $C_1 = \varrho$, $C_2 = 1$, $C_3 = \varrho$, $C_4 = \varrho^2$, $C_5 = \varrho$, $C_6 = 1$ and for $S_k = S_k(a)$ we have $S_1 = 1$, $S_2 = 1 + \varrho$, $S_3 = 2 + \varrho$, $S_4 = 2 + 2\varrho$, $S_5 = 2 + 2\varrho + \varrho^2$. Inequality (6.2) is satisfies if $\varrho \ge 0$ (from $S_2 \le S_1 + S_1$) and if $\varrho \le 1$ (from $S_2 + S_2 \ge S_1 + S_3$). So for $\frac{1}{3} \le a \le \frac{1}{2}$ it holds (6.3), and the solution $\varphi$ of (1.13) is subadditive according to Theorem 6.2. In particular, Cantor's function ($a = \frac{1}{2}$) is subadditive, cf. also [22, Section 3.2.4], [10].

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# A common generalization of the postman, radial, and river metrics

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ABSTRACT. By using a metric $d$ on a set $X$, a function $\varphi$ of $X$ to itself, a metric $\rho$ on the range of $\varphi$, and a suitable relation $\Gamma$ on $X^2$ to $X$, we construct a metric $d_{\rho\varphi\Gamma}$ on $X$. This compound metric includes the postman, radial, and river metrics as some very particular cases.

Our construction here closely follows a former one of M. Borkowski, D. Bugajewski, and H. Przybycień. Moreover, it may also be compared to that of A. G. Aksoy and B. Maurizi. However, instead of a metric projection and a collinearity relation we use the above mentioned $\varphi$ and $\Gamma$.

KEY WORDS. Generalized metrics and collinearity relations, postman, radial, and river metrics

## Introduction

The defining axioms of a metric were abstracted from the well-known properties of the Euclidean distances by M. Fréchet in 1906. The appropriateness of weakening and strengthening of these axioms have later been justified by several authors.

However, in the present paper, we shall adhere to the original axioms. Though, most distance functions occurring in analysis are *extended-valued pseudo-metrics*. Moreover, *semimetrics*, *quasi-metrics*, *ultrametrics*, and *partial metrics* have also several applications.

Thus, now a metric on a set $X$ is a function $d$ of $X^2$ to $\mathbb{R}$ such that, for any $x, y, z \in X$, we have

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(1) $d(x, y) \geq 0$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$;

(4) $d(x, y) = 0$ if and only if $x = y$.

Here, (1) and (4) are referred to as the *positive definiteness*, (2) as *symmetry*, and (3) as *the triangle inequality*. Note that, by (3) and (2), we always have

$$d(x, x) \leq d(x, y) + d(y, x) = 2\,d(x, y).$$

Hence, by using the "*equality implies indistancy*" part of (4), we can infer (1). However, it is usually more convenient to stress *nonnegativity* as a separate axiom.

For any $x, y \in X$, by defining

$$d(x, y) = \begin{cases} 0 & \text{if} \quad x = y, \\ 1 & \text{if} \quad x \neq y, \end{cases}$$

we can at once get an ultrametric $d$ on $X$. This is called the *discrete metric* on $X$. Thus, each set can be considered as a discrete metric space. Therefore, the notions of a set and a metric space are actually equivalent.

However, to provide several genuine illustrating examples for a metric, it is best to assume that $X = \mathbb{C}$ with $\mathbb{C} = \mathbb{R}^2$. Thus, for any $x, y \in X$, we may write

$$x = (x_1, x_2), \qquad \bar{x} = (x_1, -x_2);$$
$$x + y = (x_1 + y_1, \ x_2 + y_2), \qquad x\,y = (x_1 y_1 - x_2 y_2, \ x_1 y_2 + x_2 y_1).$$

Now, each $r \in \mathbb{R}$ can be identified with $(r, 0) \in X$. And each $x \in X$ can be written in the form $x = x_1 + i\,x_2$ with $i = (0, 1)$.

Moreover, for any $x, y \in X$, we may also write

$$d(x, y) = |\,x - y\,| \qquad \text{with} \qquad |\,z\,| = (z\,\bar{z})^{1/2} = \left(z_1^2 + z_2^2\right)^{1/2}.$$

Now, by using the above operations on complex numbers, it can be easily seen that $|\ |$ is a norm on $X$, and thus $d$ is a metric on $X$. This $d$ is called the Euclidean metric on $X$.

More generally, for any $x, y \in X$ and $p \in [1, \infty]$, we may also naturally define

$$d_p(x, y) = |\,x - y\,|_p \qquad \text{with} \qquad |\,z\,|_p = \begin{cases} \left(|\,z_1|^p + |\,z_2|^p\right)^{1/p} & \text{if} \quad p < \infty, \\ \max\{\,|\,z_1|, \ |\,z_2|\,\} & \text{if} \quad p = \infty. \end{cases}$$

Now, it is somewhat more difficult to prove that $|\ |_p$ is a norm on $X$, and thus $d_p$ is a metric on $X$. In particular $d_1$ and $d_\infty$ are called the *taxicab* and *supremum metrics* on $X$, respectively.

Beside the latter two extreme metrics, there are some further curious, but important metrics on $X$. For instance, for any $x, y \in X$, we may also define

$$\alpha(x, y) = \begin{cases} 0 & \text{if} \quad x = y\,, \\ |x| + |y| & \text{if} \quad x \neq y\,; \end{cases}$$

$$\beta(x, y) = \begin{cases} |x - y| & \text{if} \quad x_1 y_2 = x_2 y_1\,, \\ |x| + |y| & \text{if} \quad x_1 y_2 \neq x_2 y_1\,; \end{cases}$$

and

$$\gamma(x, y) = \begin{cases} |x_2 - y_2| & \text{if} \quad x_1 = y_1\,, \\ |x_2| + |x_1 - y_1| + |y_2| & \text{if} \quad x_1 \neq y_1\,. \end{cases}$$

Thus, by considering several cases, it can be shown that $\alpha$, $\beta$ and $\gamma$ are metrics on $X$. These are usually called the *postman, radial* and *river metrics* on $X$, respectively. (See, for instance, [17, p. 155] and [5, p. 315].) Sometimes, the radial metric is also called the *hedgehog* or *French railroad metric.*

In the present paper, following the ideas of Borkowski, Bugajewski and Przybycień [3], we construct a common generalization of the postman, radial and river metrics. Our generalization here may also be compared to that of Aksoy and Maurizi [1]. However, instead of a metric projection and collinearity relation we shall use some more general objects.

More concretely, by assuming that $d$ is a metric on a set $X$, $\varphi$ is a function of $X$ to itself, and $\rho$ is a metric on the range $\varphi[X]$ of $\varphi$, we define a generalized metric $d_{\rho\varphi}$ on $X$ such that

$$d_{\rho\varphi}(x, y) = d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), y\big)$$

for all $x, y \in X$. Moreover, by assuming that $\Gamma$ is a suitable relation on $X^2$ to $X$, we define a generalized equivalence relation $Q_{\varphi\Gamma}$ on $X$ such that

$$Q_{\varphi\Gamma} = \big\{ (x, y) \in X^2 : \quad \varphi(x) = \varphi(y) \in \Gamma(x, y) \big\}\,.$$

Thus, by defining

$$d_{\rho\varphi\Gamma}(x, y) = \begin{cases} d(x, y) & \text{if} \quad (x, y) \in Q_{\varphi\Gamma}\,, \\ d_{\rho\varphi}(x, y) & \text{if} \quad (x, y) \notin Q_{\varphi\Gamma}\,, \end{cases}$$

we can get a metric $d_{\rho\varphi\Gamma}$ on $X$ which includes the postman, radial and river metrics as some very particular cases.

For instance, the postman metric $\alpha$ can be immediately obtained from $d_{\varphi\rho\Gamma}$, by letting $d$ to be the Euclidean metric on $X = \mathbb{C}$, and defining

$$\varphi(x) = 0 \qquad \text{and} \qquad \Gamma(x, y) = \begin{cases} X & \text{if} \quad x = y\,, \\ \{0\}^c & \text{if} \quad x \neq y\,. \end{cases}$$

for all $x, y \in X$. While, to get the radial metric $\beta$, we have to consider the relation $\Gamma$ defined such that, for all $x, y \in X$, we have $\Gamma(x, y) = X$ if $x = y$, and

$$\Gamma(x, y) = \big\{ z \in X : \quad \exists\ \lambda \in K : \quad z = \lambda x + (1 - \lambda) y \big\} \qquad \text{if} \qquad x \neq y\,.$$

The latter relation $\Gamma$ can also be applied to a similar derivation of the river metric $\gamma$ by the function $\varphi$ defined such that $\varphi(x) = x_1$ for all $x \in X$.

## 1 Fixed points and equivalence relations

**Notation 1.1** *Let $X$ be a set and $\varphi$ be a function of $X$ to itself. Define*

$$A_\varphi = \big\{ x \in X : \quad \varphi(x) = x \big\}, \qquad B_\varphi = \big\{ x \in X : \quad \varphi(x) \in A_\varphi \big\},$$

$$D_\varphi = \big\{ (x, x) : \quad x \in A_\varphi \big\}, \qquad E_\varphi = \big\{ (x, y) \in X^2 : \quad \varphi(x) = \varphi(y) \big\}.$$

**Remark 1.2** Thus, $A_\varphi$ and $B_\varphi$ are the families of all fixed and idempotent points of $\varphi$, respectively.

Moreover, $D_\varphi$ is the identity function of $A_\varphi$ and $E_\varphi$ is the equivalence relation on $X$ generated by $\varphi$.

**Remark 1.3** For the identity function $\Delta_X$ of $X$, we have

$$A_{\Delta_X} = B_{\Delta_X} = X \qquad \text{and} \qquad D_{\Delta_X} = E_{\Delta_X} = \Delta_X.$$

Moreover, if $A_\varphi = X$, or equivalently $D_\varphi = \Delta_X$, then we have $\varphi = \Delta_X$.

Simple reformulations of the above definitions yield the following theorems.

**Theorem 1.4** *For any $x, y \in X$, the following assertions are equivalent:*

(1) $(x, y) \in D_\varphi$;

(2) $x, y \in A_\varphi$ *and* $x = y$;

(3) $x, y \in A_\varphi$ *and* $(x, y) \in E_\varphi$;

(4) $x = \varphi(x)$, $\quad \varphi(x) = \varphi(y)$, $\quad \varphi(y) = y$.

**Theorem 1.5** *We have*

(1) $B_\varphi = \varphi^{-1}[\,A_\varphi\,] = \{\, x \in X : \quad \varphi^2(x) = \varphi(x) \,\}$ ;

(2) $D_\varphi = \Delta_{A_\varphi} = A_\varphi^2 \cap E_\varphi$ ; (3) $E_\varphi = \varphi^{-1} \circ \varphi$ ;

(4) $\varphi[\,A_\varphi\,] \subset A_\varphi \subset \varphi[\,X\,]$ ; (5) $A_\varphi = A_{\varphi^2} \cap B_\varphi$ .

**Proof:** By the corresponding definitions, for any $x \in X$, we have

$$x \in B_\varphi \iff \varphi(x) \in A_\varphi \iff x \in \varphi^{-1}[\,A_\varphi\,],$$

$$\begin{aligned} x \in B_\varphi \iff \varphi(x) \in A_\varphi \iff \varphi(\,\varphi(x)) = \varphi(x) \\ \iff (\,\varphi \circ \varphi\,)(x) = \varphi(x) \iff \varphi^2(x) = \varphi(x)\,. \end{aligned}$$

Therefore, (1) is true.

By the corresponding definitions, it is clear that $D_\varphi = \Delta_{A_\varphi}$. Moreover, from Theorem 1.4, we can see that

$$(x,\,y) \in D_\varphi \iff (x,\,y) \in A_\varphi^2\,, \quad (x,\,y) \in E_\varphi \iff (x,\,y) \in A_\varphi^2 \cap E_\varphi\,.$$

Therefore, (2) is true. On the other hand, by the corresponding definitions, we also have

$$\begin{aligned} (x,\,y) \in E_\varphi \iff \varphi(y) = \varphi(x) \iff y \in \varphi^{-1}(\,\varphi(x)) \\ \iff y \in (\,\varphi^{-1} \circ \varphi\,)(x) \iff (x,\,y) \in \varphi^{-1} \circ \varphi\,. \end{aligned}$$

Therefore, (3) is also true.

Furthermore, we can also easily see that

$$x \in A_\varphi \implies x = \varphi(x) \implies x \in \varphi[\,X\,],$$

$$x \in A_\varphi \implies \varphi(x) = x \implies \varphi(\varphi(x)) = \varphi(x) = x \implies \varphi^2(x) = x \implies x \in A_{\varphi^2}\,,$$

$$x \in A_\varphi \implies \varphi(x) = x \implies \varphi(\varphi(x)) = \varphi(x) \implies \varphi(x) \in A_\varphi \implies x \in B_\varphi\,.$$

Hence, we can infer that

$$A_\varphi \subset \varphi[\,X\,], \qquad A_\varphi \subset X_{\varphi^2} \qquad \text{and} \qquad \varphi[\,A_\varphi\,] \subset A_\varphi\,, \qquad A_\varphi \subset B_\varphi\,.$$

Therefore, (4) and $A_\varphi \subset A_{\varphi^2} \cap B_\varphi$ is also true.

Now, to prove (5), it remains to note only that

$$\begin{aligned} x \in A_{\varphi^2} \cap B_\varphi \implies x \in A_{\varphi^2}\,,\ x \in B_\varphi \\ \implies \varphi^2(x) = x\,, \quad \varphi^2(x) = \varphi(x) \implies \varphi(x) = x \implies x \in A_\varphi\,, \end{aligned}$$

and thus $A_{\varphi^2} \cap B_\varphi \subset A_\varphi$ is also true.

## 2 Projections, involutions, and injections

**Definition 2.1** In the sequel, we shall say that:

(1) $\varphi$ is a *projection* if $\varphi^2 = \varphi$;

(2) $\varphi$ is an *involution* if $\varphi^2 = \Delta_X$;

(3) $\varphi$ is an *injection* if $\varphi$ is injective.

**Remark 2.2** Hence, it is clear that if $\varphi$ is an involution, then $\varphi$ is, in particular, also an injection.

Namely, if $x, y \in X$ such that $\varphi(x) = \varphi(y)$, then by the corresponding definitions we also have $x = \varphi^2(x) = \varphi(\varphi(x)) = \varphi(\varphi(y)) = \varphi^2(y) = y$.

**Remark 2.3** Moreover, by the corresponding definitions, it is clear that the function $\varphi$ is simultaneously both a projection and an involution if and only if $\varphi = \Delta_X$.

Now, in addition to Remark 1.3 and Theorem 1.5, we can also easily establish the following three theorems.

**Theorem 2.4** *The following assertions are equivalent:*

(1) $\varphi$ *is an injection;* (2) $\varphi^{-1}$ *is a function;* (3) $E_\varphi = \Delta_X$.

**Proof:** If $(x, y) \in E_\varphi$, then $\varphi(x) = \varphi(y)$. Hence, if (1) holds, then we can infer that $x = y$, and thus $(x, y) \in \Delta_X$. Therefore, $E_\varphi \subset \Delta_X$. Thus, by the reflexivity of $E_\varphi$, (3) also holds.

The converse implication is even more obvious. Namely, if $x, y \in X$ such that $\varphi(x) = \varphi(y)$, then $(x, y) \in E_\varphi$. Hence, if (3) holds, then we can infer that $(x, y) \in \Delta_X$, and thus $x = y$. Therefore, (1) also holds.

**Remark 2.5** If $\varphi$ is an injection, then in addition to the above assertions we can also state that $A_\varphi = B_\varphi$.

Namely, by Theorem 1.5, we always have $A_\varphi \subset B_\varphi$. Moreover, if $x \in B_\varphi$, then $\varphi(\varphi(x)) = \varphi(x)$. Hence, by the injectivity of $\varphi$, it follows that $\varphi(x) = x$, and thus $x \in A_\varphi$. Therefore, $B_\varphi \subset A_\varphi$, and thus the required equality is also true.

However, the equality $A_\varphi = B_\varphi$ does not, in general, imply the injectivity of $\varphi$.

**Example 2.6** Namely, if for instance $X = \{0, 1, 2, 3\}$, and $\varphi(0) = 0$, $\varphi(1) = 2$ and $\varphi(2) = \varphi(3) = 1$, then we have $A_\varphi = B_\varphi = \{0\}$, despite that $\varphi$ is not injective.

**Theorem 2.7** *The following assertions are equivalent:*

(1) $\varphi$ *is an involution;* (2) $\varphi = \varphi^{-1}$*;* (3) $A_{\varphi^2} = X$.

**Proof:** Now, to prove the equivalence of (1) and (3), it is enough to note only that, by Remark 1.3, we have $\varphi^2 = \Delta_X$ if and only if $A_{\varphi^2} = X$.

**Remark 2.8** If $\varphi$ is an involution, then by Remarks 2.2, 2.5 and Theorem 2.4 we also have $A_\varphi = B_\varphi$ and $E_\varphi = \Delta_X$.

However, the latter equalities do not, in general, imply that $\varphi$ is an involution.

**Example 2.9** Namely, if for instance $X = \mathbb{R}$ and $\varphi(x) = x/(1+|x|)$ for all $x \in X$, then it is clear that $A_\varphi = B_\varphi = \{0\}$. Moreover, it can be easily seen that $\varphi$ is an injection of $X$ onto $]-1,1[$ such that $\varphi^{-1}(y) = y/(1-|y|)$ for all $y \in ]-1,1[$. Thus, by the above theorems, $E_\varphi = \Delta_X$, but $\varphi$ is not an involution.

**Theorem 2.10** *The following assertions are equivalent:*

(1) *$\varphi$ is a projection;* (2) $X = B_\varphi$; (3) $A_\varphi = \varphi[X]$.

**Proof:** By the corresponding definitions and Theorem 1.5, it is clear that

$$\varphi^2 = \varphi \iff \forall\ x \in X: \quad \varphi^2(x) = \varphi(x)$$
$$\iff \forall\ x \in X: \quad x \in B_\varphi \iff X \subset B_\varphi \iff X = B_\varphi.$$

and

$$\varphi^2 = \varphi \iff \forall\ x \in X: \quad \varphi(\varphi(x)) = \varphi(x)$$
$$\iff \forall\ x \in X: \quad \varphi(x) \in A_\varphi \iff \varphi[X] \subset A_\varphi \iff A_\varphi = \varphi[X].$$

Therefore, the required equivalences are also true.

**Remark 2.11** If $\varphi$ is a projection, then by Theorems 1.5 and 2.10 we also have $A_\varphi = A_{\varphi^2} \cap B_\varphi = A_{\varphi^2} \cap X = A_{\varphi^2}$.

However, the equality $A_\varphi = A_{\varphi^2}$ does not, in general, imply that $\varphi$ is a projection even if $\varphi$ is injective.

**Example 2.12** Namely, if for instance $X$ and $\varphi$ are as in Example 2.9, then it can be easily seen that $\varphi^2(x) = x/(1+2|x|)$ for all $x \in X$. Therefore, $A_{\varphi^2} = \{0\}$ also holds.

**Remark 2.13** In this respect, it is also worth noticing that if in particular $\varphi$ is an involution such that $A_\varphi = A_{\varphi^2}$, then by Theorem 2.7 we also have $A_\varphi = X$. Therefore, by Remark 1.3, $\varphi = \Delta_X$. Thus, in particular $\varphi$ is a projection.

## 3 Weak partial pseudo-metrics specified by $\varphi$

**Definition 3.1** A function $d$ of $X^2$ to $\mathbb{R}$ is called a *$\varphi$–metric* on $X$ if for any $x, y, z \in X$ we have

(1) $d(x, y) \geq 0$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, z) \leq d(x, y) + d(y, z) - d(y, y)$;

(4) $d(x, y) = 0$ if and only if $(x, y) \in D_\varphi$.

**Remark 3.2** A function $d$ of $X^2$ to $\mathbb{R}$ satisfying conditions (1)–(3) has formerly been called a *weak partial pseudo-metric* by Heckmann [6]. This is a straightforward generalization of the *partial metric* of Matthews [11].

Now, a function $d$ of $X^2$ to $\mathbb{R}$ may be briefly called a partial metric on $X$ if it is a weak partial pseudo-metric on $X$ such that

(1) $d(x, x) \leq d(x, y)$ for all $x, y \in X$

(2) $d(x, x) = d(x, y) = d(y, y)$ implies $x = y$.

**Remark 3.3** Non-zero self-distances were already considered in a 1985 thesis of Matthews. And, the modified triangle inequality (3) was already suggested to Matthews by Wickers [21] in 1987. However, the present definition of a partial metric was only first investigated in the later works [11] and [12].

Partial metrics, being a minimal generalization of metrics allowing non-zero self-distances, were motivated by experience from theoretical computer science. The interested reader can get a rapid overview on the subject by consulting the works [13], [4] and [10], where convincing illustrating examples are also given.

Now, analogously to Propositions 2.2 and 2.4 of Heckmann [6], we can also easily establish the following two theorems.

**Theorem 3.4** *For any function $d$ of $X^2$ to $\mathbb{R}$, the following assertions are equivalent:*

(1) *$d$ is a metric on $X$;*

(2) *$d$ is a $\Delta_X$–metric on $X$;*

(3) *$d$ is a $\varphi$–metric on $X$, for some $\varphi$, such that $d(x, x) = 0$ for all $x \in X$.*

**Proof:** Since $D_{\Delta_X} = \Delta_X$, it is clear that (1) $\Longleftrightarrow$ (2) $\Longrightarrow$ (3). Moreover, if (3) holds, then by Definition 3.1 (4) we can see that $\Delta_X \subset D_\varphi$. Hence, since $D_\varphi = \Delta_{A_\varphi}$, it is clear that $A_\varphi = X$. Therefore, $\varphi = \Delta_X$, and thus (2) also holds.

**Theorem 3.5** *If $p$ is a $\varphi$–metric on $X$, then for any $x, y \in X$ we have*

(1) $d(x, x) + d(y, y) \leq 2\, d(x, y)$ ;

(2) $\min\big\{ d(x, x),\ d(y, y)\big\} \leq d(x, y)$ ;

(3) $d(x, y) = \min\limits_{z \in X} \big( d(x, z) + d(z, y) - d(z, z) \big)$ .

**Proof:** By Definition 3.1 (3) and (2), we have

$$d(x, x) \leq d(x, y) + d(y, x) - d(y, y) = 2\, d(x, y) - d(y, y),$$

and thus (1) is true. Hence, it is clear that either

$$d(x, x) \leq d(x, y) \qquad \text{or} \qquad d(y, y) \leq d(x, y).$$

Therefore, (2) also holds.

Finally, to prove (3), we need only note that, by Definition 3.1 (4), we have

$$d(x, y) \leq d(x, z) + d(z, y) - d(z, z)$$

for all $z \in X$. Moreover, we also have $d(x, y) = d(x, y) + d(y, y) - d(y, y)$.

**Remark 3.6** Note that the "small self-distances condition" in Remark 3.2 (1) can also be reformulated by writing that

$$d(x, x) = \min_{y \in X} d(x, y)$$

for all $x \in X$. Therefore, the interesting partial metric axioms are about minima.

**Remark 3.7** However, the *ultra-metric triangle inequality* [20] says that

$$d(x, z) \leq \max\big\{ d(x, y),\ d(y, z)\big\}$$

for all $x, y, z \in X$. This is also called the non-Archimedean triangle inequality.

While, the famous *four-point property* [1] says that

$$d(x, y) + d(z, w) \leq \max\big\{ d(x, z) + d(y, w),\ d(x, w) + d(y, z) \big\}$$

for all $x, y, z, w \in X$. This is closely related to the ultra-metric triangle inequality.

Note that, under the usual symmetry condition and the zero self-distances assumption, "the ultra-metric triangle inequality" implies "the four-point property" implies "the ordinary triangle inequality".

Moreover, the ordinary triangle inequality is equivalent to the *rectangle inequality* which says that

$$|\, d(x, y) - d(z, w)\,| \ \leq\ d(x, z) + d(y, w)$$

for all $x, y, z, w \in X$. There is a curious *rectrangular inequality* in [2] too.

## 4 $\varphi$-metrics derived from ordinary metrics by $\varphi$

**Notation 4.1** *Let $d$ and $\rho$ be metrics on $X$ and $\varphi[X]$, respectively. Moreover, for any $x, y \in X$, define*

$$d_{\rho\varphi}(x, y) = d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), y\big).$$

**Remark 4.2** Thus, in particular, we have

$$d_{\rho\Delta_X}(x, y) = d(x, x) + \rho(x, y) + d(y, y) = \rho(x, y)$$

for all $x, y \in X$. Therefore, $d_{\rho\Delta_X} = \rho$.

Our former definitions are mainly motivated by the following

**Theorem 4.3** *The function $d_{\rho\varphi}$ is a $\varphi$–metric on $X$ such that:*

(1) $d_{\rho\varphi}(x, y) = \rho(x, y)$ *for all* $x, y \in A_\varphi$;

(2) $d_{\rho\varphi}(x, x) = 2\, d\big(x, \varphi(x)\big)$ *for all* $x \in X$.

**Proof:** By Notation 4.1, it is clear that $d_{\rho\varphi}$ is a nonnegative, real-valued function of $X^2$. Moreover, if $x, y \in X$, then by using the nonegativity and separating properties of $d$ and $\rho$, and Theorem 1.4, we can easily see that

$$\begin{aligned} d_{\rho\varphi}(x, y) = 0 \quad &\Longleftrightarrow \quad d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), y\big) = 0 \\ &\Longleftrightarrow \quad d\big(x, \varphi(x)\big) = 0, \quad \rho\big(\varphi(x), \varphi(y)\big) = 0, \quad d\big(\varphi(y), y\big) = 0 \\ &\Longleftrightarrow \quad x = \varphi(x), \quad \varphi(x) = \varphi(y), \quad \varphi(y) = y \quad \Longleftrightarrow \quad (x, y) \in D_\varphi. \end{aligned}$$

Furthermore, by the symmetry properties of $d$ and $\rho$, it is clear that

$$\begin{aligned} d_{\rho\varphi}(y, x) &= d\big(y, \varphi(y)\big) + \rho\big(\varphi(y), \varphi(x)\big) + d\big(\varphi(x), x\big) \\ &= d\big(\varphi(y), y\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(x, \varphi(x)\big) \\ &= d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), y\big) = d_{\rho\varphi}(x, y) \end{aligned}$$

for all $x, y \in X$. Thus, $d_{\rho\varphi}$ is also symmetric.

Moreover, if $x \in X$, then by the symmetry of $d$, it is clear that

$$d_{\rho\varphi}(x, x) = d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(x)\big) + d\big(\varphi(x), x\big) = 2\, d\big(x, \varphi(x)\big).$$

Now, if $x, y, z \in X$, then by using the triangle inequality for $\rho$ and the symmetry of $d$, we can easily see that

$$\begin{aligned} d_{\rho\varphi}(x, z) &= d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(z)\big) + d\big(\varphi(z), z\big) \\ &\leq d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + \rho\big(\varphi(y), \varphi(z)\big) + d\big(\varphi(z), z\big) \\ &= d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), y\big) \\ &\quad + d\big(y, \varphi(y)\big) + \rho\big(\varphi(y), \varphi(z)\big) + d\big(\varphi(z), z\big) - 2\, d\big(y, \varphi(y)\big) \\ &= d_{\rho\varphi}(x, y) + d_{\rho\varphi}(y, z) - d_{\rho\varphi}(y, y)\,. \end{aligned}$$

Thus, we have proved that $d_{\rho\varphi}$ is a $\varphi$–metric on $X$ such that (2) holds.

Finally, if $x, y \in A_\varphi$, then we can also easily see that

$$\begin{aligned} d_{\rho\varphi}(x, y) &= d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), y\big) \\ &= d(x, x) + \rho(x, y) + d(y, y) = \rho(x, y)\,, \end{aligned}$$

and thus (1) also holds.

From (1) in Theorem 4.3, by Theorem 2.10 and Remarks 2.5 and 2.8, it is clear that in particular we have the following two corollaries.

**Corollary 4.4** *If in particular $\varphi$ is a projection, then $d_{\rho\varphi}(x, y) = \rho(x, y)$ for all $x, y \in \varphi[X]$.*

**Corollary 4.5** *If in particular $\varphi$ is an injection (involution), then $d_{\rho\varphi}(x, y) = \rho(x, y)$ for all $x, y \in B_\varphi$.*

**Notation 4.6** In the sequel, we shall simply write $d_\varphi$ in place of $d_{\rho\varphi}$ whenever $\rho$ is the restriction of $d$ to $\varphi[X]^2$.

**Remark 4.7** Thus, for any $x, y \in X$, we have

$$d_\varphi(x, y) = d\big(x, \varphi(x)\big) + d\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), y\big)\,.$$

Moreover, Theorem 4.3 and Corollaries 4.4 and 4.5 can also be specialized to $d_\varphi$.

## 5 Some further properties of the derived $\varphi$–metrics

The following two theorems will show that $d_{\rho\varphi}$ is, in general, only a weak partial pseudo-metric on $X$.

**Theorem 5.1** *For any $x, y \in X$, the following assertions are equivalent:*

(1) $d_{\rho\varphi}(x,\,x) \le d_{\rho\varphi}(x,\,y)$ ;

(2) $d\big(x,\,\varphi(x)\big) - d\big(y,\,\varphi(y)\big) \le \rho\big(\,\varphi(x),\,\varphi(y)\big)$ .

**Proof:** By Theorem 4.3 (2) and Notation 4.1, we have

$$
\begin{aligned}
& d_{\rho\varphi}(x,\,x) \le d_{\rho\varphi}(x,\,y) \\
& \qquad \iff \quad 2\,d\big(x,\,\varphi(x)\big) \le d\big(x,\,\varphi(x)\big) + \rho\big(\,\varphi(x),\,\varphi(y)\big) + d\big(\,\varphi(y),\,y\,\big) \\
& \qquad\qquad \iff \quad d\big(x,\,\varphi(x)\big) - d\big(y,\,\varphi(y)\big) \le \rho\big(\,\varphi(x),\,\varphi(y)\big)\,.
\end{aligned}
$$

Hence, by the symmetries of $d_{\rho\varphi}$ and $\rho$, we can immediately derive

**Corollary 5.2** *The following assertions are equivalent:*

(1) $d_{\rho\varphi}(x,\,x) \le d_{\rho\varphi}(x,\,y)$ *for all* $x,\,y \in X$;

(2) $\big|\,d\big(x,\,\varphi(x)\big) - d\big(y,\,\varphi(y)\big)\,\big| \le \rho\big(\,\varphi(x),\,\varphi(y)\big)$ *for all* $x,\,y \in X$.

**Theorem 5.3** *For any* $x,\,y \in X$, *the following assertions are equivalent:*

(1) $d_{\rho\varphi}(x,\,x) = d_{\rho\varphi}(x,\,y) = d_{\rho\varphi}(y,\,y)$ ;

(2) $\varphi\,(x) = \varphi(y)$ *and* $d\big(x,\,\varphi(x)\big) = d\big(y,\,\varphi(y)\big)$ .

**Proof:** Quite similarly, as in the proof of Theorem 5.1, we can see that

$$
\begin{aligned}
& d_{\rho\varphi}(x,\,x) = d_{\rho\varphi}(x,\,y) = d_{\rho\varphi}(y,\,y) \\
& \qquad \iff \quad \rho\big(\,\varphi(x),\,\varphi(y)\big) = d\big(x,\,\varphi(x)\big) - d\big(y,\,\varphi(y)\big) = 0 \\
& \qquad\qquad \iff \quad \varphi\,(x) = \varphi(y)\,, \quad d\big(x,\,\varphi(x)\big) = d\big(y,\,\varphi(y)\big)\,.
\end{aligned}
$$

Hence, it is clear that in particular we also have

**Corollary 5.4** *If in particular* $\varphi$ *is an injection, then* $d_{\rho\varphi}(x,x) = d_{\rho\varphi}(x,y) = d_{\rho\varphi}(y,y)$ *implies* $x = y$ .

**Remark 5.5** Thus, if $\varphi$ is an injection, then by Theorem 4.3 and Corollary 5.4 $d_{\rho\varphi}$ is a weak partial metric on $X$ in the sense of Heckmann [6].

The following example shows that $d_{\varphi}$ need not be a partial metric even if in particular $\varphi$ is an involution on $X$.

**Example 5.6** If for instance $X = \mathbb{C}$, $\varphi(x) = \bar{x}$ for all $x \in X$, and $d$ is the Euclidean metric on $X$, then $\varphi$ is an involution on $X$ such that

$$\begin{aligned} d\big(\varphi(1),\, \varphi(i)\big) &= |\,1 - (-i\,)\,| = \sqrt{2} \\ &< 2 = \big|\,|\,1-1\,| - |\,i - (-i\,)\,|\,\big| = \big|\,d\big(1,\, \varphi(1)\big) - d\big(i,\, \varphi(i)\big)\,\big|\,. \end{aligned}$$

Thus, by Remark 5.5 and Corollary 5.2, $d_{\varphi}$ is a weak partial metric, but not a partial metric on $X$.

In addition Theorem 4.3, we can also easily prove the following

**Theorem 5.7** *If in particular* $d(u,\, v) \leq \rho(u,\, v)$ *for all* $u,\, v \in \varphi\,[\,X\,]$, *then for any* $x,\, y \in X$ *we have*

(1) $d(x,\, y) \leq d_{\rho\varphi}(x,\, y)\,;$ (2) $d_{\rho\varphi}(x,\, x) \leq d_{\rho\varphi}(x,\, y) + d(x,\, y)\,.$

**Proof:** By using the triangle inequality for $d$ and the assumption of the theorem, we can easily see that

$$\begin{aligned} d(x,\, y) &\leq d\big(x,\, \varphi(x)\big) + d\big(\,\varphi(x),\; y\,\big) \\ &\leq d\big(x,\, \varphi(x)\big) + d\big(\,\varphi(x),\; \varphi(y)\big) + d\big(\,\varphi(y),\; y\,\big) \\ &\leq d\big(x,\, \varphi(x)\big) + \rho\big(\,\varphi(x),\; \varphi(y)\big) + d\big(\,\varphi(y),\; y\,\big) = d_{\rho\varphi}(x,\, y)\,. \end{aligned}$$

Moreover, by using Theorem 4.3 (2), the symmetry of $d$, the triangle inequality for $d$, and the assumption of the theorem, we can also easily see that

$$\begin{aligned} d_{\rho\varphi}(x,\, x) &= 2\, d\big(x,\, \varphi(x)\big) = d\,\big(x,\, \varphi(x)\big) + d\big(\varphi(x),\, x\big) \\ &\leq d\big(x,\, \varphi(x)\big) + d\big(\,\varphi(x),\; \varphi(y)\big) + d\big(\,\varphi(y),\, x\big) \\ &\leq d\big(x,\, \varphi(x)\big) + \rho\big(\,\varphi(x),\; \varphi(y)\big) + d\big(\,\varphi(y),\, y\big) + d(y,\, x) \\ &= d_{\rho\varphi}(x,\, y) + d(x,\, y)\,. \end{aligned}$$

From this theorem, according to Notation 4.6, we can immediately derive

**Corollary 5.8** *For any any* $x,\, y \in X$ *we have*

(1) $d(x,\, y) \leq d_{\varphi}(x,\, y)\,;$ (2) $d_{\varphi}(x,\, x) \leq d_{\varphi}(x,\, y) + d(x,\, y)\,.$

## 6 $\varphi$–dominated, $\varphi$–equivalence relations

**Definition 6.1** A relation $Q$ on $X$ is called *$\varphi$–transitive* if

$$(x, y) \in Q\,, \quad (y, z) \in Q\,, \quad y \in A_{\varphi}^{c} \quad \Longrightarrow \quad (x, z) \in Q\,.$$

**Remark 6.2** Note that thus every transitive relation $Q$ on $X$ is, in particular, $\varphi$–transitive.

Moreover, if in particular $A_{\varphi} = \emptyset$, then every $\varphi$–transitive relation $Q$ on $X$ is already transitive.

While, if in particular $A_{\varphi} = X$, or equivalently $\varphi = \Delta_X$, then every relation $Q$ on $X$ is $\varphi$–transitive.

By using the composition and box products of relations, we can easily establish some concise characterizations of $\varphi$–transitive relations.

**Definition 6.3** For any two relations $F$ and $G$ on $X$, the relation $F \boxtimes G$ on $X^2$, defined such that

$$\big(F \boxtimes G\big)(x, y) = F(x) \times G(y)$$

for all $x, y \in X$, is called the *box product* of the relations $F$ and $G$.

**Remark 6.4** Note that, in contrast to the composition, the box product of two relations can be easily extended to arbitrary family of relations.

However, in the sequel we shall only need the box product of two relations which is closely related to the composition of relations by the following

**Lemma 6.5** *If $F$ and $G$ are relations on $X$, then for any $A \subset X^2$ we have*

$$\big(F \boxtimes G\big)[A] = G \circ A \circ F^{-1}.$$

**Proof:** If $(z, w) \in \big(F \boxtimes G\big)[A]$, then by the corresponding definitions there exists $(x, y) \in A$ such that $(z, w) \in \big(F \boxtimes G\big)(x, y)$, and thus $(z, w) \in F(x) \times G(y)$. Hence, we can infer that $z \in F(x)$ and $w \in G(y)$, and thus $(x, z) \in F$ and $(y, w) \in G$. Now, by using that $(z, x) \in F^{-1}$ and $(x, y) \in A$, we can see that $(z, y) \in A \circ F^{-1}$. Hence, by using that $(y, w) \in G$, we can already see that $(z, w) \in G \circ \big(A \circ F^{-1}\big)$. This shows that $\big(F \boxtimes G\big)[A] \subset G \circ \big(A \circ F^{-1}\big)$.

The converse inclusion can be proved quite similarly. Hence, by the associativity of the composition, it is clear that the required equality can also be stated.

**Remark 6.6** From the above lemma, we can immediately infer that

$$\big( F \boxtimes G \big)(x,\, y) = G \circ \{(x,\, y)\} \circ F^{-1}$$

for all $x,\, y \in X$. Moreover, by taking $F^{-1}$ in place of $F$, we can also see that

$$G \circ F = G \circ \Delta_X \circ F = \big( F^{-1} \boxtimes G \big) [\, \Delta_X \,] \,.$$

Therefore, the composition and the box products are actually equivalent tools.

However, it is now more important to note that, by using Lemma 6.5, we can also easily prove the following

**Theorem 6.7** *For a relation $Q$ on $X$, the following assertions are equivalent:*

(1) $Q$ *is $\varphi$-transitive;*

(2) $Q \circ \Delta_{A_\varphi^c} \circ Q \subset Q\,;$ (3) $\big( Q^{-1} \boxtimes Q \big) [\, \Delta_{A_\varphi^c} \,] \subset Q\,.$

**Proof:** Note that

$$\begin{aligned} (x,\, z) \in Q^{-1}(y) \times Q(y) \quad &\Longrightarrow \quad x \in Q^{-1}(y)\,, \quad z \in Q(y) \\ &\Longrightarrow \quad y \in Q(x)\,, \quad z \in Q(y) \quad \Longrightarrow \quad (x,\, y) \in Q\,, \quad (y,\, z) \in Q \end{aligned}$$

for all $x,\, y,\, z \in X$. Therefore, if (1) holds, then we have

$$\big( Q^{-1} \boxtimes Q \big)(y,\, y) = Q^{-1}(y) \times Q(y) \subset Q$$

for all $y \in A_\varphi^c$. Hence, it is clear that

$$\big( Q^{-1} \boxtimes Q \big) [\, \Delta_{A_\varphi^c} \,] = \bigcup \big\{ \big( Q^{-1} \boxtimes Q \big)(y,\, y) : \quad y \in A_\varphi^c \big\} \subset Q\,,$$

and thus (3) also holds.

The converse implication (3) $\Longrightarrow$ (1) can be proved quite similarly. Moreover, by using Lemma 6.5, we can see that

$$\big( Q^{-1} \boxtimes Q \big) [\, \Delta_{A_\varphi^c} \,] = Q \circ \Delta_{A_\varphi^c} \circ Q\,.$$

Therefore, inclusions (2) and (3) are also equivalent.

**Remark 6.8** Note that, under the notation $\Theta_\varphi = X \times A_\varphi^c$, we have

$$\big( \Delta_{A_\varphi^c} \circ Q \big)(x) = \Delta_{A_\varphi^c} [\, Q(x) ] = Q(x) \cap A_\varphi^c = Q(x) \cap \Theta_\varphi(x) = \big( Q \cap \Theta_\varphi \big)(x)$$

for all $x \in X$. Therefore, $\Delta_{A_\varphi^c} \circ Q = Q \cap \Theta_\varphi$ is also true.

From Theorem 6.7, we can immediately get the nontrivial part of the following

**Corollary 6.9** *For a relation $Q$ on $X$, the following assertions are equivalent:*

(1) $Q$ *is transitive;*

(2) $Q \circ Q \subset Q$; (3) $\big(Q^{-1} \boxtimes Q\big)\,[\,\Delta_X\,] \subset Q$.

**Proof:** If $X$ is not a singleton, then by the Axiom of Choice there exists a function $\varphi$ of $X$ to itself such that $\varphi(x) \in X \setminus \{x\}$ for all $x \in X$, and thus $A_\varphi = \emptyset$. Therefore, Theorem 6.7 can be applied.

While, if in particular $X$ is a singleton, then $Q_0 = \emptyset$ and $Q_1 = X^2$ are the only relations on $X$. Moreover, these two extreme relations trivially satisfy conditions (1)–(3) even if $X$ is not a singleton.

**Definition 6.10** A tolerance (reflexive and symmetric) relation $Q$ on $X$ is called a *$\varphi$–equivalence* if it is $\varphi$–transitive.

**Remark 6.11** Quite similarly, an intolerance (reflexive and antisymmetric) relation $Q$ on $X$ may be called a *$\varphi$-partial order* if it is $\varphi$-transitive.

In the sequel, we shall also need the following

**Definition 6.12** A relation $Q$ on $X$ is called *$\varphi$-dominated* if $Q \subset E_\varphi$. (That is, $(x, y) \in Q$ implies $\varphi(x) = \varphi(y)$.)

**Remark 6.13** Note that if in particular $\varphi$ is injective or equivalently $E_\varphi = \Delta_X$, then $\Delta_X$ is the only $\varphi$–dominated reflexive relation on $X$.

Moreover, note that $\Delta_X$ is actually an equivalence relation on $X$ such that $\Delta_X \subset E_\varphi$. Thus, in particular, it is a $\varphi$–dominated $\varphi$–equivalence relation on $X$ for any function $\varphi$ of $X$ to itself.

## 7 A metric derived from $d$ and $d_{\rho\varphi}$ by $Q$

**Notation 7.1** *In addition to Notation 1.1 and 4.1, assume now that $Q$ is a $\varphi$–dominated, $\varphi$–equivalence relation on $X$.*

*Moreover, for any $x, y \in X$, define*

$$d_{\rho\varphi Q}(x, y) = \begin{cases} d(x, y) & \text{if} \quad (x, y) \in Q, \\ d_{\rho\varphi}(x, y) & \text{if} \quad (x, y) \notin Q. \end{cases}$$

**Remark 7.2** Note that if in particular $\varphi = \Delta_X$, then by Remarks 4.2 and 6.13 we have $d_{\rho\varphi} = \rho$ and $Q = \Delta_X$.

Therefore, in this particular case, $d_{\rho\varphi Q}(x, y) = d(x, y)$ for $(x, y) \in \Delta_X$ and $d_{\rho\varphi Q}(x, y) = \rho(x, y)$ for $(x, y) \notin \Delta_X$. Hence, since $d(x, y) = 0 = \rho(x, y)$ if $(x, y) \in \Delta_X$, we can already see that $d_{\rho\varphi Q} = \rho$.

Our former definitions are mainly motivated by the following

**Theorem 7.3** *The function $d_{\rho\varphi Q}$ is a metric on $X$ such that*

(1) $d_{\rho\varphi Q}(x, y) = \rho(x, y)$ *for all* $x, y \in A_\varphi$*;*

(2) $d(x, y) \leq d_{\rho\varphi Q}(x, y)$ *for all* $x, y \in X$ *whenever* $d(u, v) \leq \rho(u, v)$ *for all* $u, v \in \varphi[X]$.

**Proof:** For the sake of brevity, define $\delta = d_{\rho\varphi}$ and $\sigma = d_{\rho\varphi Q}$. Then by the corresponding definitions, for any $x, y \in X$, we have

$$\sigma(x, y) = \begin{cases} d(x, y) & \text{if} \quad (x, y) \in Q, \\ \delta(x, y) & \text{if} \quad (x, y) \notin Q, \end{cases}$$

with

$$\delta(x, y) = d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), y\big)$$

Moreover, by Theorems 4.3, 1.4 and 5.7, $\delta$ is a symmetric, nonnegative, real-valued function of $X^2$, satisfying the triangle inequality, such that

(a) $\delta(x, y) = \rho(x, y)$ for all $x, y \in A_\varphi$;

(b) $\delta(x, y) = 0$ is equivalent to $x = \varphi(x) = \varphi(y) = y$ for all $x, y \in X$;

(c) $d(x, y) \leq \delta(x, y)$ for all $x, y \in X$ whenever $d(u, v) \leq \rho(u, v)$ for all $u, v \in \varphi[X]$.

Now, from the definition of $\sigma$, it is clear that $\sigma$ is a nonnegative, real-valued function of $X^2$. Moreover, if $x \in X$, then from the reflexivity of $Q$ we can at once see that $(x, x) \in Q$, and thus $\sigma(x, x) = d(x, x) = 0$.

On the other hand, if $x, y \in X$ such that $\sigma(x, y) = 0$, then in the case $(x, y) \in Q$ we can see that $d(x, y) = \sigma(x, y) = 0$, and thus $x = y$. While, in the case $(x, y) \notin Q$ we can see that $\delta(x, y) = \sigma(x, y) = 0$, and thus $x = y$. Therefore, $\sigma(x, y) = 0$ if and only if $x = y$.

Moreover, if $x, y \in X$, then in the case $(x, y) \in Q$ we can see that $(y, x) \in Q$, and thus $\sigma(x, y) = d(x, y) = d(y, x) = \sigma(y, x)$. While, in the case $(x, y) \notin Q$ we can see that $(y, x) \notin Q$, and thus $\sigma(x, y) = \delta(x, y) = \delta(y, x) = \sigma(y, x)$. Therefore, $\sigma$ is also symmetric function of $X^2$.

On the other hand, if $x, y \in A_\varphi$, then in the case $(x, y) \in E_\varphi$, we can see that $x = \varphi(x) = \varphi(y) = y$. Hence, by the reflexivity of $Q$, it follows that $(x, y) \in Q$. Therefore, $\sigma(x, y) = d(x, y) = 0 = \rho(x, y)$ because of $x = y$. While, in the case $(x, y) \notin E_\varphi$, we can see that $(x, y) \notin Q$. Therefore, $\sigma(x, y) = \delta(x, y) = \rho(x, y)$ is also true by (a). This proves (1).

Moreover, if $d(u, v) \leq \rho(u, v)$ for all $u, v \in \varphi[X]$, then by (c) we have $d(x, y) \leq \delta(x, y)$ for all $x, y \in X$. Hence, since $\sigma(x, y)$ is either $d(x, y)$ or $\delta(x, y)$, it is clear that $\sigma(x, y) \leq \delta(x, y)$ also holds for all $x, y \in X$. Therefore, (2) is also true.

Now, to complete the proof, it remains only to prove that $\sigma$ also satisfies the triangle inequality. This nontrivial fact will be proved in the next section by considering several cases.

From (1) in Theorem 7.3, by Theorem 2.10 and Remarks 2.5 and 2.8, it is clear that in particular we have the following two corollaries.

**Corollary 7.4** *If in particular $\varphi$ is a projection, then $d_{\rho\varphi Q}(x, y) = \rho(x, y)$ for all $x, y \in \varphi[X]$.*

**Corollary 7.5** *If in particular $\varphi$ is an injection (involution), then $d_{\rho\varphi Q}(x, y) = \rho(x, y)$ for all $x, y \in B_\varphi$.*

**Notation 7.6** *In the sequel, analogously to Notation 4.6, we shall simply write $d_{\varphi Q}$ in place of $d_{\rho\varphi Q}$ whenever $\rho$ is the restriction of $d$ to $\varphi[X]^2$.*

**Remark 7.7** Thus, for any $x, y \in X$, we have

$$d_{\varphi Q}(x, y) = \begin{cases} d(x, y) & \text{if} \quad (x, y) \in Q, \\ d_\varphi(x, y) & \text{if} \quad (x, y) \notin Q. \end{cases}$$

Moreover, Theorem 7.3 can be specialized in the following form.

**Theorem 7.8** *The function $d_{\varphi Q}$ is a metric on $X$ such that*

(1) $d_{\varphi Q}(x, y) = \rho(x, y)$ *for all* $x, y \in A_\varphi$;

(2) $d(x, y) \leq d_{\varphi Q}(x, y)$ *for all* $x, y \in X$.

## 8 The proof of the triangle inequality for $\sigma = d_{\rho\varphi Q}$

To complete the proof of Theorem 7.3, it has remained to show that, for any $x, y, z \in X$, we have

$$\sigma(x, z) \leq \sigma(x, y) + \sigma(y, z).$$

For this, according to definition of $\sigma$ and the positions of the points $(x, z), (x, y)$ and $(y, z)$ with respect to $Q$, we have to consider several cases.

Note that if each of the above tree points is in $Q$, then by the definition of $\sigma$ and the triangle inequality for $d$ we evidently have

$$\sigma(x, z) = d(x, z) \leq d(x, y) + d(y, z) = \sigma(x, y) + \sigma(y, z).$$

While, if none of the above three points is in $Q$, then by the definition of $\sigma$ and the triangle inequality for $\delta$ we evidently have

$$\sigma(x, z) = \delta(x, z) \leq \delta(x, y) + \delta(y, z) = \sigma(x, y) + \sigma(y, z).$$

Assume now that $(x, z) \notin Q$, but $(x, y)$, $(y, z) \in Q$. Then, by $Q \subset E_\varphi$ and the definition of $E_\varphi$, we have $\varphi(x) = \varphi(y) = \varphi(z)$. Moreover, by the $\varphi$–transitivity of $Q$, we also have $y \notin A_\varphi^c$. Therefore, by the definition of $A_\varphi$, we have $\varphi(y) = y$, and thus also $\varphi(x) = y$ and $\varphi(z) = y$. Hence, by the definitions of $\sigma$ and $\delta$ and the zero self-distance property of $\rho$, it is clear that

$$\begin{aligned}\sigma(x, z) &= \delta(x, z) = d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(z)\big) + d\big(\varphi(z), z\big)\\ &= d(x, y) + \rho(y, y) + d(y, z) = d(x, y) + d(y, z) = \sigma(x, y) + \sigma(y, z).\end{aligned}$$

Therefore, instead of the required inequality, the corresponding equality is also true.

Next, assume that $(x, z) \in Q$, $(x, y) \notin Q$, but $(y, z) \in Q$. Then, by $Q \subset E_\varphi$ and the definition of $E_\varphi$, we have $\varphi(x) = \varphi(z)$ and $\varphi(y) = \varphi(z)$. Moreover, by the symmetry of $Q$, we also have $(z, y) \in Q$. Hence, by the $\varphi$–transitivity of $Q$, it is clear that $z \notin A_\varphi^c$. Therefore, by the definition of $A_\varphi$, we have $\varphi(z) = z$, and thus also $\varphi(x) = z$ and $\varphi(y) = z$. Now, by the definitions of $\sigma$ and $\delta$, and the zero self-distance property of $\rho$ and the symmetry of $d$, we can see that

$$\begin{aligned}\sigma(x, y) &= \delta(x, y) = d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), y\big)\\ &= d(x, z) + \rho(z, z) + d(z, y) = d(x, z) + d(y, z) = \sigma(x, z) + \sigma(y, z),\end{aligned}$$

and hence

$$\sigma(x, z) = \sigma(x, y) - \sigma(y, z) = \sigma(x, y) + \sigma(y, z) - 2\,\sigma(y, z).$$

Therefore, by the nonnegativity of $\sigma$, the required inequality is also true.

Quite similarly, if $(x, z)$, $(x, y) \in Q$, but $(y, z) \notin Q$, then by $Q \subset E_\varphi$ and the definition of $E_\varphi$ we can see that $\varphi(x) = \varphi(z)$ and $\varphi(x) = \varphi(y)$. Moreover, by the symmetry of $Q$, we also have $(y, x) \in Q$. Hence, by the $\varphi$–transitivity of $Q$, it is clear that $x \notin A_\varphi^c$. Therefore, by the definition of $A_\varphi$, we have $\varphi(x) = x$, and thus also $\varphi(y) = x$ and $\varphi(z) = x$. Now, by the definitions of $\sigma$ and $\delta$, and the symmetry of $d$ and the zero self-distance property of $\rho$, we can see that

$$\begin{aligned}\sigma(y, z) = \delta(y, z) &= d\big(y, \varphi(y)\big) + \rho\big(\varphi(y), \varphi(z)\big) + d\big(\varphi(z), z\big) \\ &= d(y, x) + \rho(x, x) + d(x, z) = d(x, y) + d(x, z) = \sigma(x, y) + \sigma(x, z),\end{aligned}$$

and hence

$$\sigma(x, z) = \sigma(y, z) - \sigma(x, y) = \sigma(x, y) + \sigma(y, z) - 2\,\sigma(x, y).$$

Therefore, by the nonnegativity of $\sigma$, the required inequality is also true.

To continue the proof, assume now that $(x, z)$, $(x, y) \notin Q$, but $(y, z) \in Q$. Then, by $Q \subset E_\varphi$ and the definition of $E_\varphi$, we have $\varphi(y) = \varphi(z)$. Moreover, by the definitions of $\sigma$ and $\delta$, and the triangle inequality for $d$, we can see that

$$\begin{aligned}\sigma(x, z) = \delta(x, z) &= d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(z)\big) + d\big(\varphi(z), z\big) \\ &= d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), z\big) \\ &\leq d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(y)\big) + d\big(\varphi(y), y\big) + d(y, z) \\ &= \delta(x, y) + d(y, z) = \sigma(x, y) + \sigma(y, z).\end{aligned}$$

Therefore, the required inequality is again true.

Quite similarly, if $(x, z) \notin Q$, $(x, y) \in Q$, but $(y, z) \notin Q$, then by $Q \subset E_\varphi$ and the definition of $E_\varphi$ we have $\varphi(x) = \varphi(y)$. Moreover, by the definitions of $\sigma$ and $\delta$, and the triangle inequality for $d$, we can see that

$$\begin{aligned}\sigma(x, z) = \delta(x, z) &= d\big(x, \varphi(x)\big) + \rho\big(\varphi(x), \varphi(z)\big) + d\big(\varphi(z), z\big) \\ &= d\big(x, \varphi(y)\big) + \rho\big(\varphi(y), \varphi(z)\big) + d\big(\varphi(z), z\big) \\ &\leq d(x, y) + d\big(y, \varphi(y)\big) + +\rho\big(\varphi(y), \varphi(z)\big) + d\big(\varphi(z), z\big) \\ &= d(x, y) + \delta(y, z) = \sigma(x, y) + \sigma(y, z).\end{aligned}$$

Therefore, the required inequality is again true.

Finally, to complete the proof, assume now that $(x, z) \in Q$, but $(x, y) \notin Q$ and $(y, z) \notin Q$. Then, by $Q \subset E_\varphi$ and the definition of $E_\varphi$, we have $\varphi(x) = \varphi(z)$. Moreover, by the definitions of $\sigma$ and $\delta$, the zero self-distance property of $\rho$, the triangle inequality for $d$ and $\rho$, and the nonnegativity of $d$, we can see that

$$\begin{aligned}
\sigma(x,\,z) = d(x,\,z) = d(x,\,z) + \rho\big(\varphi(x),\,\varphi(z)\big) \\
\le d\big(x,\,\varphi(x)\big) + d\big(\varphi(x),\,z\big) + \rho\big(\varphi(x),\,\varphi(y)\big) + \rho\big(\varphi(y),\,\varphi(z)\big) \\
= d\big(x,\,\varphi(x)\big) + \rho\big(\varphi(x),\,\varphi(y)\big) + \rho\big(\varphi(y),\,\varphi(z)\big) + d\big(\varphi(z),\,z\big) \\
\le d\big(x,\,\varphi(x)\big) + \rho\big(\varphi(x),\,\varphi(y)\big) + d\big(\varphi(y),\,y\big) \\
+ d\big(y,\,\varphi(y)\big) + \rho\big(\varphi(y),\,\varphi(z)\big) + d\big(\varphi(z),\,z\big) \\
= \delta(x,\,y) + \delta(y,\,z) = \sigma(x,\,y) + \sigma(y,\,z)\,.
\end{aligned}$$

Therefore, the required inequality is again true.

## 9 Collinearity-like relations

To construct $\varphi$–dominated, $\varphi$–equivalence relations on $X$, we shall need the following assumptions.

**Notation 9.1** *Suppose that $\Gamma$ is a relation on $X^2$ to $X$ such that, for any $x,\,y,\,z \in X$, we have*

(1) $\Gamma(x,\,x) = X$ ;

(2) $\Gamma(x,\,y) = \Gamma(y,\,x)$ ;

(3) $\Gamma(x, y) \cap \Gamma(y,\,z) \cap \{y\}^c \subset \Gamma(x,\,z)$ .

**Remark 9.2** Note that, to guarantee property (2), it is enough to assume only that $\Gamma(x,\,y) \subset \Gamma\,(y,\,x)$ for all $x,\,y \in X$ with $x \neq y$ .

**Remark 9.3** While, to guarantee property (3), it is enough to assume only that $w \in \Gamma(x, y)$ and $w \in \Gamma(y,\,z)$ imply $w \in \Gamma(x,\,z)$ for all $x,\,y,\,z \in X$ with $x \neq y$ , $y \neq z$ and $y \neq w$ .

Namely, if $x = y$ , then $w \in \Gamma(y,\,z)$ already implies that $w \in \Gamma(x,\,z)$ . While, if $y = z$ , then $w \in \Gamma(x, y)$ already implies that $w \in \Gamma(x,\,z)$ .

**Remark 9.4** Moreover, it is also worth noticing that if (1) is already assumed, then we may also suppose that $x \neq z$ .

Namely, if $x = z$ , then by (1) we have $\Gamma(x,\,z) = X$ , and thus $w \in \Gamma(x,\,z)$ trivially holds.

**Definition 9.5** If $\Gamma$ is as in Notation 9.1, then we say that $\Gamma$ is a *pre-collinearity relation* for $X$.

While, if $\Gamma$ is a pre-collinearity relation for $X$ such that for any $x,\,y,\,z \in X$

(4) $z \in \Gamma(x, y) \implies x \in \Gamma(y, z)$,

then we say that $\Gamma$ is a *collinearity relation* for $X$.

**Remark 9.6** Note that, if (1) is already assumed, then to guarantee (4) it is enough to suppose only that $z \in \Gamma(x, y)$ implies $x \in \Gamma(y, z)$ for all $x, y, z \in X$ with $y \neq z$.

Namely, if $y = z$, then by (1) we have $\Gamma(y, z) = X$, and thus $x \in \Gamma(y, z)$ trivially holds.

**Remark 9.7** While, if (1), (2) and (4) are already assumed, then to guarantee (3) it is enough to suppose only that $w \in \Gamma(x, y)$ and $w \in \Gamma(y, z)$ imply $w \in \Gamma(x, z)$ for any four, pairwise distinct points $x, y, z$ and $w$ of $X$.

Namely, if $x = w$, then by (1) we have $\Gamma(w, x) = X$, and thus $z \in \Gamma(w, x)$ trivially holds. Hence, by (4), it follows that $w \in \Gamma(x, z)$. While, if $z = w$, then by (1) we have $\Gamma(w, z) = X$, and thus $x \in \Gamma(w, z)$ trivially holds. Hence, by (4) and (2), it follows that $w \in \Gamma(z, x) = \Gamma(x, z)$.

**Remark 9.8** In connection with hypothesis (4), it is also worth noticing that if $z \in \Gamma(x, y)$, then by (2) we also have $z \in \Gamma(y, x)$. Hence, by (4), we can infer that $y \in \Gamma(x, z)$. Thus, by (2), the inclusion $y \in \Gamma(z, x)$ also holds.

**Example 9.9** For any $x, y \in X$, define $\Gamma(x, y) = X$. Then, $\Gamma$ is a collinearity relation for $X$ such that instead of property (3) we actually have $\Gamma(x, y) \cap \Gamma(y, z) = \Gamma(x, z)$ for all $x, y, z \in X$.

**Example 9.10** Let $0$ be a fixed element of $X$, and for any $x, y \in X$ define

$$\Gamma(x, y) = \begin{cases} X & \text{if} \quad x = y\,, \\ \{0\}^c & \text{if} \quad x \neq y\,. \end{cases}$$

Then, $\Gamma$ is a pre-collinearity relation on $X$ such that instead of property (3) we actually have $\Gamma(x, y) \cap \Gamma(y, z) \subset \Gamma(x, z)$ for all $x, y, z \in X$.

**Proof:** Now, properties (1) and (2) are trivially satisfied. Moreover, if $w \in \Gamma(x, y)$ and $w \in \Gamma(y, z)$, then we can easily see that $w \in \Gamma(x, z)$.

Namely, if $w \neq 0$, then $w \in \{0\}^c \subset \Gamma(x, z)$. While, if $w = 0$, then $w \in \Gamma(x, y)$ and $w \in \Gamma(y, z)$ imply that $x = y$ and $y = z$. Hence, it follows that that $x = z$, and thus $\Gamma(x, z) = X$. Therefore, $w \in \Gamma(x, z)$ trivially holds.

**Remark 9.11** Note that now we actually have $\Gamma(x, y) \cap \Gamma(y, z) = \Gamma(x, z)$ for any three, pairwise distinct points $x$, $y$ and $z$ of $X$.

However, if for instance $x, y \in X$ such that $x \neq y$, then in contrast to Example 9.9 we have $\Gamma(x, y) \cap \Gamma(y, x) = \{0\}^c \neq X = \Gamma(x, x)$.

**Remark 9.12** Moreover, it is also worth noticing that if $y, z \in \{0\}^c$ such that $y \neq z$, then $z \in \{0\}^c = \Gamma(0, y)$, but $0 \notin \{0\}^c = \Gamma(y, z)$.

Therefore, in contrast to Example 9.9 and our forthcoming examples, the relation $\Gamma$ considered in Example 9.10 is not, in general, a collinearity relation for $X$.

## 10 Two more natural examples for collinearity relations

**Example 10.1** For any $x, y \in X$, define

$$\Gamma(x, y) = \begin{cases} X & \text{if} \quad x = y\,, \\ \{x, y\} & \text{if} \quad x \neq y\,. \end{cases}$$

Then, $\Gamma$ is collinearity relation for $X$.

**Proof:** Properties 9.1 (1) and (2) are again trivially satisfied. Moreover, if $w \in \Gamma(x, y)$ and $w \in \Gamma(y, z)$ such that $x \neq y$, $y \neq z$ and $y \neq w$, then we can note that $w = x$ and $w = z$. Hence, it follows that $x = z$, and thus $\Gamma(x, z) = X$. Therefore, $w \in \Gamma(x, z)$ trivially holds. Hence, by Remark 9.3, we can see that 9.1 (3) also holds.

Finally, to prove property 9.5(4), we can note that if $x = y$, then $x \in \{y, z\} \subset \Gamma(y, z)$. While, if $x \neq y$, then $z \in \Gamma(x, y)$, with $z \neq y$, implies that $z = x$. Therefore, $x \in \{y, z\} = \Gamma(y, z)$. Thus, by Remark 9.6, property 9.5 (4) also holds.

**Remark 10.2** Note that if $x$, $y$ and $z$ are pairwise distinct points of $X$, then

$$y \in \{x, y\} = \Gamma(x, y) \quad \text{and} \quad y \in \{y, z\} = \Gamma(y, z)\,, \quad \text{but} \quad y \notin \{x, z\} = \Gamma(x, z)\,.$$

Therefore, in contrast to Example 9.9 and Remark 9.11, we now have $\Gamma(x, y) \cap \Gamma(y, z) \not\subset \Gamma(x, z)$.

**Example 10.3** Let $X$ be a vector space over $K$, and for any $x, y \in X$, define $\Gamma(x, x) = X$ if $x = y$, and

$$\Gamma(x, y) = \big\{ z \in X : \quad \exists\ \lambda \in K : \quad z = \lambda x + (1 - \lambda) y \big\} \qquad \text{if} \qquad x \neq y\,.$$

Then, $\Gamma$ is a collinearity relation for $X$.

**Proof:** If $z \in \Gamma(x, y)$ such that $x \neq y$, then there exists $\lambda \in K$ such that $z = \lambda x + (1 - \lambda) y$. Hence, by taking $\mu = 1 - \lambda$, we can see that

$$z = (1 - \lambda) y + \lambda x = \mu y + (1 - \mu) x\,.$$

Therefore, $z \in \Gamma(y, x)$ also holds. This shows that $\Gamma(x, y) \subset \Gamma(y, x)$ whenever $x \neq y$. Thus, by Remark 9.2, property 9.1 (2) also holds.

Moreover, if $\lambda \neq 0$, then by taking $\nu = 1 - 1/\lambda$, we can see that

$$x = \big( 1 - 1/\lambda \big)\, y + \big( 1/\lambda \big)\, z = \nu\, y + (1 - \nu)\, z\,.$$

Therefore, $x \in \Gamma(y, z)$. Now, to see that property 9.5 (4) also holds, it remains to note only that if $\lambda = 0$, then $z = y$. Therefore, $\Gamma(y, z) = X$, and thus $x \in \Gamma(y, z)$ trivially holds.

Finally, to prove property 9.1 (3), note that if $w \in \Gamma(x, y) \cap \Gamma(y, z) \cap \{y\}^c$, then

$$w \in \Gamma(x, y)\,, \qquad w \in \Gamma(y, z) \qquad \text{and} \qquad y \neq w\,.$$

Moreover, by Remarks 9.3 and 9.4, we may suppose that $x \neq y$, $y \neq z$ and $x \neq z$ also hold. Now, by the above assumptions, we can state that there exist $\lambda$, $\mu \in K$ such that

$$w = \lambda\, x + (1 - \lambda)\, y \qquad \text{and} \qquad w = \mu\, y + (1 - \mu)\, z$$

Hence, by using that $y \neq w$ and $x \neq z$, we can infer that $\lambda \neq 0$ and $\lambda + \mu \neq 1$.

Namely, if $\lambda + \mu = 1$, then we also have

$$w = (1 - \lambda)\, y + \lambda\, z\,, \qquad \text{and thus} \qquad \lambda\, x + (1 - \lambda)\, y = (1 - \lambda)\, y + \lambda\, z\,.$$

Hence, we can infer that $\lambda\, x = \lambda\, z$, and thus $x = z$ since $\lambda \neq 0$. And this contradicts the assumption that $x \neq z$.

From the above equations on $w$, we can also infer that

$$\mu\, w = \lambda\, \mu\, x + (1 - \lambda)\mu\, y \qquad \text{and} \qquad (1 - \lambda)\, w = (1 - \lambda)\, \mu\, y + (1 - \lambda)(1 - \mu)\, z\,,$$

and thus

$$(\lambda + \mu - 1)\, w = \mu\, w - (1 - \lambda)\, w = \lambda\, \mu\, x - (1 - \lambda)(1 - \mu)\, z = (\lambda + \mu - 1 - \lambda\, \mu)\, z\,.$$

Hence, since $\lambda + \mu \neq 1$, we can already see that

$$w = \frac{\lambda\, \mu}{\lambda + \mu - 1}\, x + \left( 1 - \frac{\lambda\, \mu}{\lambda + \mu - 1} \right) z\,.$$

Therefore, $w \in \Gamma(x, z)$, and thus property 9.5 (4) also holds.

**Remark 10.4** Note that if in particular $X = K$, then for any $x, y, z \in X$, with $x \neq y$, we have

$$z = \frac{z - y}{x - y}\, x + \left( 1 - \frac{z - y}{x - y} \right) y\,.$$

Therefore, in this particular case $\Gamma(x, y) = X$ also holds for all $x, y \in X$ with $x \neq y$.

**Remark 10.5** However, if in particular $X = \mathbb{C}$ and $K = \mathbb{R}$, then we have

$$1 = 0 \cdot 0 + (1-0)\,1, \qquad 1 = 1 \cdot 1 + (1-1)\,i, \qquad \text{and} \qquad 1 \neq \lambda 0 + (1-\lambda)\,i$$

for all $\lambda \in K$. Therefore,

$$1 \in \Gamma(0,\,1), \qquad 1 \in \Gamma(1,\,i), \qquad \text{but} \qquad 1 \notin \Gamma(0,\,i).$$

Thus, in contrast to Example 9.10 and Remark 9.11, we now have $\Gamma(0,1) \cap \Gamma(1,i) \not\subset \Gamma(0,i)$. This show that the set $\{y\}^c$ cannot be omitted from assumption 9.1 (3).

## 11 A $\varphi$–dominated, $\varphi$-equivalence defined by $\varphi$ and $\Gamma$

**Notation 11.1** *Define*

$$Q_{\varphi\,\Gamma} = \big\{ (x,\,y) \in E_\varphi : \quad \varphi(x) \in \Gamma(x,\,y) \big\}.$$

**Remark 11.2** Hence, because of $\Gamma\,(x,\,x) = X$, it is clear that

$$(x,\,y) \in Q_{\Delta_X \Gamma} \iff x = y, \quad x \in \Gamma(x,\,y) \iff x = y.$$

Therefore, in particular we have $Q_{\Delta_X \Gamma} = \Delta_X$.

Our former definitions have been mainly motivated by the following

**Theorem 11.3** *$Q_{\varphi\,\Gamma}$ is a $\varphi$–dominated, $\varphi$–equivalence relation on $X$.*

**Proof:** If $x \in X$, then because of $(x,\,x) \in E_\varphi$ and $\varphi(x) \in X = \Gamma(x,\,x)$ we also have $(x,\,x) \in Q_{\varphi\,\Gamma}$. Therefore, $Q_{\varphi\,\Gamma}$ is also reflexive on $X$.

Moreover, if $(x,\,y) \in Q_{\varphi\,\Gamma}$, then $(x,\,y) \in E_\varphi$ and $\varphi(x) \in \Gamma(x,\,y)$. Hence, by the symmetry of $E_\varphi$ and $\Gamma$ and the definition of $E_\varphi$, we can see that $(y,\,x) \in E_\varphi$ and $\varphi(y) = \varphi(x) \in \Gamma(x,\,y) = \Gamma(y,\,x)$ also hold. Therefore, $(y,\,x) \in Q_{\varphi\,\Gamma}$, and thus $Q_{\varphi\,\Gamma}$ is also symmetric.

Now, since $Q_{\varphi\,\Gamma}$ is evidently $\varphi$-dominated, it remains to prove only that $Q_{\varphi\,\Gamma}$ is $\varphi$–transitive too. For this, assume that $(x,\,y) \in Q_{\varphi\,\Gamma}$, $(y,\,z) \in Q_{\varphi\,\Gamma}$ and $y \in A_\varphi^c$. Then, by the corresponding definitions, we have

$$(x,\,y) \in E_\varphi, \qquad \varphi(x) \in \Gamma(x,\,y) \qquad \text{and} \qquad (y,\,z) \in E_\varphi, \qquad \varphi(y) \in \Gamma(y,\,z),$$

and moreover $y \neq \varphi(y)$. Hence, by the transitivity of $E_\varphi$, we can infer that $(x,\,z) \in E_\varphi$. Moreover, since $\varphi(x) = \varphi(y) \in \Gamma(y,\,z)$ and $\varphi(x) = \varphi(y) \in \{y\}^c$, by using property 9.1 (3) we can also infer that $\varphi(x) \in \Gamma(x,\,z)$. Therefore, $(x,\,z) \in Q_{\varphi\,\Gamma}$ also holds.

**Corollary 11.4** *We have*

(1) $Q_{\varphi\Gamma} = \Delta_X \cup \big( Q_{\varphi\Gamma} \setminus \Delta_X \big)$ ;

(2) $Q_{\varphi\Gamma} \setminus \Delta_X = \big\{ (x, y) \in E_\varphi \setminus \Delta_X : \quad \varphi(x) \in \Gamma(x, y) \big\}$ .

**Proof:** By Theorem 11.3, we have $\Delta_X \subset Q_{\varphi\Gamma}$ , and thus

$$Q_{\varphi\Gamma} = \Delta_X \cup \big( Q_{\varphi\Gamma} \setminus \Delta_X \big) .$$

Moreover, by the corresponding definitions, it is clear that

$$\begin{aligned} Q_{\varphi\Gamma} \setminus \Delta_X &= \big\{ (x, y) \in E_\varphi : \quad \varphi(x) \in \Gamma(x, y) \big\} \setminus \Delta_X \\ &= \big\{ (x, y) \in E_\varphi \setminus \Delta_X : \quad \varphi(x) \in \Gamma(x, y) \big\} . \end{aligned}$$

**Corollary 11.5** *If $\varphi$ is an injection (involution), then $Q_{\varphi\Gamma} = \Delta_X$ .*

**Proof:** By Notation 11.1 and Theorem 2.4 (Remark 2.8), we have $Q_{\varphi\Gamma} \subset E_\varphi = \Delta_X$ . Therefore, $Q_{\varphi\Gamma} \setminus \Delta_X = \emptyset$ , and thus by Corollary 11.4 the required equality is also true.

**Example 11.6** If in particular $\Gamma$ is as in Example 9.9, then $Q_{\varphi\Gamma} = E_\varphi$ . Namely, by the corresponding definitions, we have

$$Q_{\varphi\Gamma} = \big\{ (x, y) \in E_\varphi : \quad \varphi(x) \in \Gamma(x, y) \big\} = \big\{ (x, y) \in E_\varphi : \quad \varphi(x) \in X \big\} .$$

**Example 11.7** If in particular $\Gamma$ is as in Example 9.10, then

$$Q_{\varphi\Gamma} = \Delta_X \cup \big\{ (x, y) \in E_\varphi \setminus \Delta_X : \quad \varphi(x) \neq 0 \big\} .$$

Namely, by the corresponding definitions, we have

$$\begin{aligned} Q_{\varphi\Gamma} \setminus \Delta_X &= \big\{ (x, y) \in E_\varphi \setminus \Delta_X : \quad \varphi(x) \in \Gamma(x, y) \big\} \\ &= \big\{ (x, y) \in E_\varphi \setminus \Delta_X : \quad \varphi(x) \in \{0\}^c \big\} . \end{aligned}$$

Therefore, by Corollary 11.4, the required equality is also true.

**Example 11.8** If in particular $\Gamma$ is as in Example 10.1, then

$$Q_{\varphi\Gamma} = \Delta_X \cup \big\{ (x, y) \in E_\varphi \setminus \Delta_X : \quad \varphi(x) = x \;\text{ or }\; \varphi(x) = y \big\} .$$

Namely, by the corresponding definitions, we have

$$\begin{aligned} Q_{\varphi\Gamma} \setminus \Delta_X &= \big\{ (x, y) \in E_\varphi : \quad \varphi(x) \in \Gamma(x, y) \big\} \\ &= \big\{ (x, y) \in E_\varphi \setminus \Delta_X : \quad \varphi(x) \in \{x, y\} \big\} . \end{aligned}$$

Therefore, by Corollary 11.4, the required equality is also true.

**Example 11.9** If in particular $\Gamma$ is as in Example 10.3, then

$$Q_{\varphi\Gamma} = \Delta_X \cup \big\{ (x, y) \in E_\varphi \setminus \Delta_X : \quad \exists\ \lambda \in K : \quad \varphi(x) = \lambda x + (1 - \lambda) y \big\}.$$

Namely, by the corresponding definitions, we have

$$\begin{aligned} Q_{\varphi\Gamma} \setminus \Delta_X &= \big\{ (x, y) \in E_\varphi \setminus \Delta_X : \quad \varphi(x) \in \Gamma(x, y) \big\} \\ &= \big\{ (x, y) \in E_\varphi \setminus \Delta_X : \quad \exists\ \lambda \in K : \quad \varphi(x) = \lambda x + (1 - \lambda) y \big\}. \end{aligned}$$

Therefore, by Corollary 11.4, the required equality is also true.

**Remark 11.10** Note that in the statements of above examples, we may simply write $E_\varphi$ in place $E_\varphi \setminus \Delta_X$.

## 12 Metrics derived from $d$ and $\rho$ by $Q_{\varphi\Gamma}$

**Notation 12.1** *Now, according to Theorem 11.3 and Notation 7.1, we define* $d_{\rho\varphi\Gamma} = d_{\rho\varphi Q_{\varphi\Gamma}}$.

**Remark 12.2** Thus, for any $x, y \in X$, we have

$$d_{\rho\varphi\Gamma}(x, y) = \begin{cases} d(x, y) & \text{if} \quad (x, y) \in Q_{\varphi\Gamma}, \\ d_{\rho\varphi}(x, y) & \text{if} \quad (x, y) \notin Q_{\varphi\Gamma}. \end{cases}$$

Moreover, in particular, by Theorem 11.3 and Remark 7.2, we have $d_{\rho\Delta_X\Gamma} = \rho$.

Furthermore, as an immediate consequence of Theorems 11.3 and 7.3, we can at once state the following

**Theorem 12.3** *The function $d_{\rho\varphi\Gamma}$ is a metric on $X$ such that*

(1) $d_{\rho\varphi\Gamma}(x, y) = \rho(x, y)$ *for all* $x, y \in A_\varphi$;

(2) $d(x, y) \le d_{\rho\varphi\Gamma}(x, y)$ *for all* $x, y \in X$ *whenever* $d(u, v) \le \rho(u, v)$ *for all* $u, v \in \varphi[X]$.

Now, analogously to Corollaries 7.4 and 7.5, we can also state

**Corollary 12.4** *If in particular $\varphi$ is a projection, then $d_{\rho\varphi\Gamma}(x, y) = \rho(x, y)$ for all $x, y \in \varphi[X]$.*

**Corollary 12.5** *If in particular $\varphi$ is an injection (involution), then $d_{\rho\varphi\Gamma}(x, y) = \rho(x, y)$ for all $x, y \in B_\varphi$.*

**Notation 12.6** In the sequel, analogously to Notation 7.6, we shall simply write $d_{\varphi\Gamma}$ in place of $d_{\rho\varphi\Gamma}$ whenever $\rho$ is the restriction of $d$ to $\varphi[X]^2$.

**Remark 12.7** Thus, for any $x, y \in X$, we have

$$d_{\varphi\Gamma}(x, y) = \begin{cases} d(x, y) & \text{if} \quad (x, y) \in Q_{\varphi\Gamma}, \\ d_{\varphi}(x, y) & \text{if} \quad (x, y) \notin Q_{\varphi\Gamma}. \end{cases}$$

Moreover, for instance, Theorem 12.3 can be specialized in the following form.

**Theorem 12.8** *The function $d_{\varphi\Gamma}$ is a metric on $X$ such that*

(1) $d_{\varphi\Gamma}(x, y) = \rho(x, y)$ *for all* $x, y \in A_{\varphi}$*;*

(2) $d(x, y) \leq d_{\varphi\Gamma}(x, y)$ *for all* $x, y \in X$*.*

**Notation 12.9** *In the forthcoming illustrating examples, by specializing our former notation, we shall assume that $X = \mathbb{C}$ and $d$ is the Euclidean metric on $X$.*

**Example 12.10** If $\varphi(x) = 0$ for all $x \in X$, then it is clear that $\varphi$ is a projection,

$$A_{\varphi} = \{0\}, \qquad B_{\varphi} = X \qquad \text{and} \qquad D_{\varphi} = \Delta_{\{0\}}, \qquad E_{\varphi} = X^2.$$

Moreover, if $x, y \in X$, then according to Notation 4.1 we can also easily see that

$$d_{\rho\varphi}(x, y) = d(x, 0) + \rho(0, 0) + d(0, y) = |x| + |y|.$$

Note that, by Theorem 4.3, $d_{\rho\varphi}$ is a $\varphi$–metric on $X$. Thus, according to property 3.1(4), we have $d_{\rho\varphi}(x, y) = 0$ if and only if $(x, y) \in \Delta_{\{0\}}$, i. e., $x = y = 0$. But, in contrast to property 3.1(3), the corresponding equality is also true.

Furthermore, if $\Gamma$ is as in Example 9.10, then by Example 11.7, Remark 11.10 and the definition of $\varphi$ we have

$$Q_{\varphi\Gamma} = \Delta_X \cup \left\{ (x, y) \in X^2 : \quad 0 \neq 0 \right\} = \Delta_X \cup \emptyset = \Delta_X.$$

Now, if $x, y \in X$, then by the above observations we can see that, according to Notation 12.1, we simply have

$$d_{\rho\varphi\Gamma}(x, y) = \begin{cases} 0 & \text{if} \quad x = y, \\ |x| + |y| & \text{if} \quad x \neq y. \end{cases}$$

Therefore, in the present particular case, $d_{\rho\varphi\Gamma}$ is just the postman metric on $X$ mentioned in the Introduction.

**Remark 12.11** Note that if $\Gamma$ is as in Example 9.9, then by Example 11.6 we have $Q_{\varphi\Gamma} = E_{\varphi} = X^2$. Therefore, by Notation 12.1, we have $d_{\rho\varphi\Gamma}(x, y) = d(x, y)$ for all $x, y \in X$, and thus $d_{\rho\varphi\Gamma} = d$.

**Remark 12.12** While, if $\Gamma$ is as in Example 10.1, then by Example 11.8, Remark 11.10 and the definition of $\varphi$ we have

$$Q_{\varphi\Gamma} = \Delta_X \cup \big\{ (x, y) \in X^2 : \quad 0 = x \quad \text{or} \quad 0 = y \big\} = \Delta_X \cup \{0\} \times \mathbb{R} \cup \mathbb{R} \times \{0\} .$$

Therefore, by Notation 12.1, for any $x, y \in X$ we have

$$d_{\rho\varphi\Gamma}(x, y) = \begin{cases} |x - y| & \text{if} \quad x = y \quad \text{or} \quad x y = 0 , \\ |x| + |y| & \text{if} \quad x \neq y \quad \text{and} \quad x y \neq 0 . \end{cases}$$

Therefore, in the present particular case, $d_{\rho\varphi\Gamma}$ is again the postman metric on $X$.

## 13 A similar derivation of the radial metric

Analogously to the above derivations of the postman metric, the identically zero function can also be used to derive the radial metric.

**Example 13.1** Suppose now that $\varphi$ is as in Example 12.10, but $\Gamma$ is as in Example 10.3 with $K = \mathbb{R}$. Then, in addition to the corresponding statements of Example 12.10, by Example 11.9, Remark 11.10 and the definition of $\varphi$ we have

$$Q_{\varphi\Gamma} = \Delta_X \cup \big\{ (x, y) \in X^2 : \quad \exists\ \lambda \in \mathbb{R} : \quad 0 = \lambda x + (1 - \lambda) y \big\} .$$

Next, we show that, in the present particular case, we simply have

$$Q_{\varphi\Gamma} = \big\{ (x, y) \in X^2 : \quad (x \bar{y})_2 = 0 \big\} .$$

For this, note that, if $(x, y) \in Q_{\varphi\Gamma}$, such that $x \neq y$, then there exists $\lambda \in \mathbb{R}$ such that $0 = \lambda x + (1 - \lambda - 1) y$, and thus

$$\lambda x = (\lambda - 1) y .$$

Hence, if $y \neq 0$, then we can infer that $\lambda \neq 0$, and thus

$$x = \big( 1 - 1/\lambda \big) y.$$

Now, by taking $\mu = 1 - 1/\lambda$, we can also see that

$$x \bar{y} = \mu y \bar{y} = \mu |y|^2 ,$$

and thus

$$(x\,\bar{y})_2=0 \qquad \text{and} \qquad (x\,\bar{y})_1=\mu\,|\,y\,|^{\,2}\,.$$

Thus, in particular $(\,x,\,y\,)\in Q_{\varphi\Gamma}$ implies $(x\,\bar{y})_2=0$ whenever $x\neq y$ and $y\neq 0$. Moreover, we can also note that

$$(\,x\,\bar{x})_2=\big(\,|\,x\,|^2\big)_2=0 \qquad \text{and} \qquad (\,x\,\bar{0})_2=(\,x\,0)_2=0_2=0$$

for all $x\in X$. Therefore,

$$Q_{\varphi\Gamma}\subset\big\{(x,\,y)\in X^{\,2}:\quad (\,x\,\bar{y}\,)_2=0\,\big\}\,.$$

To prove the converse inclusion, suppose now that $x,\,y\in X$ such that $(x\,\bar{y})_2=0$. Now, if $y\neq 0$, and thus $|\,y\,|\neq 0$, then by defining

$$\mu=(\,x\,\bar{y}\,)_1/|\,y\,|^{\,2}\,,$$

we can see that

$$x\,\bar{y}=(\,x\,\bar{y}\,)_1=\mu\,|\,y\,|^{\,2}=\mu\,y\,\bar{y}\,.$$

Hence, since $\bar{y}\neq 0$, we can infer that

$$x=\mu\,y\,.$$

Now, if $x\neq y$, and thus $\mu\neq 1$, then we can also see that

$$\frac{1}{1-\mu}x+\left(1-\frac{1}{1-\mu}\right)y=\frac{1}{1-\mu}\,\mu\,y+\frac{-\mu}{1-\mu}\,y=0\,.$$

Hence, we can already see that $(x,\,y)\in Q_{\varphi\Gamma}$. Thus, in particular $(x\,\bar{y})_2=0$ implies $(x,\,y)\in Q_{\varphi\Gamma}$ whenever $x\neq y$ and $y\neq 0$. Moreover, by the reflexivity of $Q_{\varphi\Gamma}$ and the equality $0\,x=(\,0-1\,)\,0$, we can also note that

$$(x,\,x)\in Q_{\varphi\Gamma} \qquad \text{and} \qquad (x,\,0)\in Q_{\varphi\Gamma}$$

for all $x\in X$. Therefore,

$$\big\{(x,\,y)\in X^{\,2}:\quad (\,x\,\bar{y}\,)_2=0\,\big\}\subset Q_{\varphi\Gamma}\,,$$

and thus the required equality is also true.

Now, if $x,\,y\in X$, then by the above observations we can see that, according to Notation 12.1, we simply have

$$d_{\rho\varphi\Gamma}(x,\,y)=\begin{cases}\;|\,x-y\,| & \text{if} \quad (x\,\bar{y})_2=0\,,\\ |\,x\,|+|\,y\,| & \text{if} \quad (x\,\bar{y})_2\neq 0\,.\end{cases}$$

Hence, by noticing that

$$(x\,\bar{y})_2 = 0 \quad \Longleftrightarrow \quad x_2\,y_1 - x_1\,y_2 = 0 \quad \Longleftrightarrow \quad x_1\,y_2 = x_2\,y_1\,,$$

we can see that, in the present particular case, $d_{\rho\varphi\Gamma}$ is just the radial metric on $X$ mentioned in the Introduction.

**Remark 13.2** Here, it is also worth mentioning that if $X = R^2$ with an integral domain $R$ and

$$Q = \big\{ (x,\,y) \in X^2 : \quad x_1\,y_2 = x_2\,y_1 \big\}\,,$$

then $Q$ is a $\varphi$–equivalence relation on $X$ with $\varphi = X \times \{0\}$.

Moreover, we have

$$Q\big((0,\,0)\big) = X, \qquad Q\big((s,\,0)\big) = R \times \{0\}\,,$$

and

$$Q\big((s,\,t)\big) = \big\{ (u,\,v) \in X : \quad s\,v = t\,u \big\} = t/s$$

for all $s,\,t \in R$ with $s \neq 0$.

Thus, in particular $Q\big((s,t)\big)$, with $s, t \in R$ and $s \neq 0$, is a maximal partial multiplier, and so also a maximal partial homomorphism on $R$ to itself. Moreover, the family $Q\,[\,\{0\}^c \times R\,]$ is the classical quotient field of $R$. (For some generalizations of the above ideas, see [18], [19], and the references therein.)

## 14 A similar derivation of the river metrics

Now, in contrast to the derivations the postman and radial metrics, the identically zero function has to be replaced by a non-constant one to derive the river metric.

**Example 14.1** If $\varphi(x) = x_1$ for all $x \in X$, then it is clear that $\varphi$ is a projection,

$$A_\varphi = \mathbb{R}\,, \quad B_\varphi = X \qquad \text{and} \qquad D_\varphi = \Delta_{\mathbb{R}}\,, \quad E_\varphi = \big\{ (x,\,y) \in X^2 : \quad x_1 = y_1 \big\}\,.$$

Moreover, if $x,\,y \in X$, then by Remark 4.7, we can also easily see that

$$d_\varphi(x,\,y) = d(x,\,x_1) + d(x_1,\,y_1) + d(y_1,\,y) = |\,x_2|\,+\,|\,x_1 - y_1\,|\,+\,|\,y_2|\,.$$

Note that, by the corresponding particular case of Theorem 4.3, $d_\varphi$ is a $\varphi$–metric on $X$. Thus, according to property 3.1(4), we have $d_\varphi(x,\,y) = 0$ if and only if $(x,\,y) \in \Delta_{\mathbb{R}}$, i.e., $x = y \in \mathbb{R}$.

Furthermore, if $\Gamma$ is as in Example 10.3, then by Example 11.9, Remark 11.10 and the definition of $\varphi$ we have

$$Q_{\varphi\Gamma} = \Delta_X \cup \{ (x, y) \in X^2 : \quad x_1 = y_1 , \quad \exists \lambda \in \mathbb{R} : \quad x_1 = \lambda x + (1 - \lambda) y \} .$$

Next, we show that, in the present particular case, we simply have

$$Q_{\varphi\Gamma} = E_\varphi = \{ (x, y) \in X^2 : \quad x_1 = y_1 \} .$$

For this, note that, by Notation 11.1, $Q_{\varphi\Gamma} \subset E_\varphi$ automatically holds. Moreover, if $(x, y) \in E_\varphi$ such that $x \neq y$, then

$$x_1 = y_1 \qquad \text{and} \qquad x_2 \neq y_2 .$$

Hence, by taking

$$\lambda = \frac{y_2}{y_2 - x_2} ,$$

we have not only

$$x_1 = \lambda x_1 + (1 - \lambda) y_1 , \qquad \text{but also} \qquad 0 = \lambda x_2 + (1 - \lambda) y_2 .$$

Therefore,

$$x_1 = \lambda x + (1 - \lambda) y ,$$

and thus $(x, y) \in Q_{\varphi\Gamma}$ also holds. Hence, by the reflexivity of $Q_{\varphi\Gamma}$, it is clear that $E_\varphi \subset Q_{\varphi\Gamma}$, and thus the required equality is also true.

Now, if $x, y \in X$, then by the above observations we can see that, according to Notation 12.6, we simply have

$$d_{\varphi\Gamma}(x, y) = \begin{cases} |x_2 - y_2| & \text{if} \quad x_1 = y_1 , \\ |x_2| + |x_1 - y_1| + |y_2| & \text{if} \quad x_1 \neq y_1 . \end{cases}$$

Therefore, in the present particular case, $d_{\varphi\Gamma}$ is just the river metric on $X$ mentioned in the Introduction.

**Remark 14.2** Note that if $\Gamma$ is as in Example 9.9, then by Example 11.6 we also have $Q_{\varphi\Gamma} = E_\varphi$. Therefore, by the above observations, $d_{\varphi\Gamma}$ is again the river metric on $X$.

**Remark 14.3** While, if $\Gamma$ is as in Example 9.10, then by Example 11.7, Remark 11.10 and the definition of $\varphi$ we have

$$Q_{\varphi\Gamma} = \Delta_X \cup \{ (x, y) \in X^2 : \quad x_1 = y_1 , \quad x_1 \neq 0 \} .$$

Therefore, by Notation 12.6, for any $x, y \in X$ we have

$$d_{\varphi\Gamma}(x, y) = \begin{cases} 0 & \text{if} \quad x = y , \\ |x_2 - y_2| & \text{if} \quad x_1 = y_1 , \quad x_1 \neq 0 , \\ |x_2| + |x_1 - y_1| + |y_2| & \text{if} \quad x_1 \neq y_1 \ \text{ or } \ x_1 = 0 , \quad x \neq y . \end{cases}$$

**Remark 14.4** Moreover, if $\Gamma$ is as in Example 10.1, then by Example 11.8, Remark 11.10 and the definition of $\varphi$ we have

$$Q_{\varphi\Gamma} = \Delta_X \cup \big\{ (x, y) \in X^2 : \quad x_1 = y_1 , \quad x \in \mathbb{R} \quad \text{or} \quad y \in \mathbb{R} \big\} .$$

Therefore, by Notation 12.6, for any $x, y \in X$ we have

$$d_{\varphi\Gamma}(x, y) = \begin{cases} 0 & \text{if} \quad x = y , \\ |\, x_2 - y_2 \,| & \text{if} \quad x_1 = y_1 , \ x \in \mathbb{R} \ \text{or} \ y \in \mathbb{R} , \\ |\, x_2| + |\, x_1 - y_1| + |\, y_2| & \text{if} \quad x_1 \neq y_1 \ \text{or} \ x, y \notin \mathbb{R} , \ x \neq y . \end{cases}$$

## 15 Some further illustrating examples

**Example 15.1** If $\varphi(x) = \bar{x}$ for all $x \in X$, then it is clear that $\varphi$ is an involution,

$$A_\varphi = B_\varphi = \mathbb{R} \qquad \text{and} \qquad D_\varphi = \Delta_{\mathbb{R}} , \qquad E_\varphi = \Delta_X .$$

Moreover, if $x, y \in X$, then according to Remark 4.7 we can also easily see that

$$d_\varphi(x, y) = |\, x - \bar{x} \,| + |\, \bar{x} - \bar{y} \,| + |\, \bar{y} - y \,| = 2\,|\, x_2| + |\, x - y \,| + 2\,|\, y_2| .$$

Note that, by the corresponding particular case of Theorem 4.3, $d_\varphi$ is a $\varphi$–metric on $X$. Thus, according to property 3.1(4), we have $d_\varphi(x, y) = 0$ if and only if $(x, y) \in \Delta_{\mathbb{R}}$ , i. e., $x = y \in \mathbb{R}$ .

Furthermore, by Corollary 11.5, we can now at once see that $Q_{\varphi\Gamma} = \Delta_X$ . Therefore, if $x, y \in X$, then by the above observations we can see that, according to Notation 12.6, we simply have

$$d_{\varphi\Gamma}(x, y) = \begin{cases} 0 & \text{if} \quad x = y , \\ |\, x - y \,| + 2\,|\, x_2| + 2\,|\, y_2| & \text{if} \quad x \neq y . \end{cases}$$

**Example 15.2** If for all $x \in X$ we have

$$\varphi(x) = \begin{cases} 0 & \text{if} \quad x = 0 , \\ 1/x & \text{if} \quad x \neq 0 , \end{cases}$$

then it is clear that $\varphi$ is an involution,

$$A_\varphi = B_\varphi = \{-1, 0, 1\} \qquad \text{and} \qquad D_\varphi = \Delta_{\{-1,0,1\}} , \qquad E_\varphi = \Delta_X .$$

Moreover, according to Remark 4.7, we can also easily see that

$$d_\varphi(0,\,0) = |\,0-0\,| + |\,0-0\,| + |\,0-0\,| = 0\,,$$
$$d_\varphi(x,\,0) = \left|\,x-\frac{1}{x}\,\right| + \left|\,\frac{1}{x}-0\,\right| + |\,0-0\,| = \frac{|\,x^2-1\,|}{|\,x\,|} + \frac{1}{|\,x\,|}\,,$$
$$d_\varphi(0,\,y) = |\,0-0\,| + \left|\,0-\frac{1}{y}\,\right| + \left|\,\frac{1}{y}-y\,\right| = \frac{1}{|\,y\,|} + \frac{|\,y^2-1\,|}{|\,y\,|}$$

and

$$d_\varphi(x,\,y) = \left|\,x-\frac{1}{x}\,\right| + \left|\,\frac{1}{x}-\frac{1}{y}\,\right| + \left|\,\frac{1}{y}-y\,\right| = \frac{|\,x^2-1\,|}{|\,x\,|} + \frac{|\,x-y\,|}{|\,x\,y\,|} + \frac{|\,y^2-1\,|}{|\,y\,|}\,,$$

for all $x,\,y\in X$ with $x\neq 0$ and $y\neq 0$.

Note that, by the corresponding particular case of Theorem 4.3, $d_\varphi$ is a $\varphi$–metric on $X$. Thus, according to property 3.1(4), we have $d_\varphi(x,\,y)=0$ if and only if $(x,\,y)\in\Delta_{\{-1,0,1\}}$, i. e., $x=y\in\{-1,\,0,\,1\}$.

Furthermore, by Corollary 11.5, we can now at once see that $Q_{\varphi\Gamma}=\Delta_X$. Therefore, if $x,\,y\in X$, then by the above observations we can see that, according to Notation 12.6, we have

$$d_{\varphi\Gamma}(x,\,y) = \begin{cases} 0 & \text{if} \quad x=y\,, \\ \frac{|\,x^2-1\,|+1}{|\,x\,|}\,, & \text{if} \quad x\neq 0\,,\quad y=0\,, \\ \frac{|\,y^2-1\,|+1}{|\,y\,|} & \text{if} \quad x=0\,,\quad y\neq 0\,, \\ \frac{|\,x^2y-y\,|+|\,x\,y^2-x\,|+|\,x-y\,|}{|\,x\,y\,|} & \text{if} \quad x\neq 0\,,\quad y\neq 0\,,\quad x\neq y\,. \end{cases}$$

**Example 15.3** If for all $x\in X$ we have $\varphi(x)=\operatorname{sgn}(x)$, i. e.,

$$\varphi(x) = \begin{cases} 0 & \text{if} \quad x=0\,, \\ x/|\,x\,| & \text{if} \quad x\neq 0\,, \end{cases}$$

then it is clear that $\varphi$ is a projection,

$$A_\varphi = \{0\}\cup S\,, \qquad B_\varphi = X \qquad \text{and} \qquad D_\varphi = \Delta_{\{0\}\cup S}\,,$$

where $S=\{\,x\in X:\ |\,x\,|=1\,\}$. Moreover, we can also easily see that

$$E_\varphi = \Delta_{\{0\}}\cup\big\{(x,\,r\,x):\quad x\in\{0\}^c,\quad r>0\,\big\}\,.$$

Namely, for any $x\,,\,y\in X$, we have

$$(x,\,y\,)\in E_\varphi \iff \varphi(x)=\varphi(y)$$
$$\iff \big(\,x=0\,,\quad y=0\,\big) \quad\text{or}\quad \big(\,x\neq 0\,,\quad y\neq 0\,,\quad x/|\,x\,|=y/|\,y\,|\big)$$
$$\iff \big(\,x=0\,,\quad y=0\,\big) \quad\text{or}\quad \big(\,x\neq 0\,,\quad y\neq 0\,,\quad y=r\,x \ \text{ with } \ r=|\,y\,|/|\,x\,|\,\big)\,.$$
$$\iff \big(\,x=0\,,\quad y=0\,\big) \quad\text{or}\quad \big(\,x\neq 0\,,\quad y=r\,x \ \text{ with } \ r>0\,\big)\,.$$

Moreover, according to Remark 4.7, we can also easily see that

$$
\begin{aligned}
d_\varphi(0,\,0) &= |\,0-0\,| + |\,0-0\,| + |\,0-0\,| = 0\,,\\
d_\varphi(x,\,0) &= \left|\,x - \frac{x}{|\,x\,|}\,\right| + \left|\,\frac{x}{|\,x\,|} - 0\,\right| + |\,0-0\,| = \big|\,|\,x\,|-1\,\big| + 1\,,\\
d_\varphi(0,\,y) &= |\,0-0\,| + \left|\,0 - \frac{y}{|\,y\,|}\,\right| + \left|\,\frac{y}{|\,y\,|} - y\,\right| = 1 + \big|\,|\,y\,|-1\,\big|\,,
\end{aligned}
$$

and

$$
\begin{aligned}
d_\varphi(x,\,y) &= \left|\,x - \frac{x}{|\,x\,|}\,\right| + \left|\,\frac{x}{|\,x\,|} - \frac{y}{|\,y\,|}\,\right| + \left|\,\frac{y}{|\,y\,|} - y\,\right|\\
&= \big|\,|\,x\,|-1\,\big| + \frac{\big|\,x\,|\,y\,| - |\,x\,|\,y\,\big|}{|\,x\,|\,|\,y\,|} + \big|\,|\,y\,|-1\,\big|\,,
\end{aligned}
$$

for all $x,\,y\in X$ with $x\neq 0$ and $y\neq 0$.

Note that, by the corresponding particular case of Theorem 4.3, $d_\varphi$ is a $\varphi$–metric on $X$. Thus, according to property 3.1(4), we have $d_\varphi(x,\,y)=0$ if and only if $(x,\,y)\in\Delta_{\{0\}\cup S}$, i. e., $x=y$ and either $y=0$ or $|\,y\,|=1$.

Furthermore, if $\Gamma$ is as in Example 9.10, then by Example 11.7, Remark 11.10, and the definition of $\varphi$ we can see that

$$
\begin{aligned}
Q_{\varphi\Gamma} &= \Delta_X \cup \big\{\,(\,x,\,y\,)\in E_\varphi : \quad \varphi(x)\neq 0\,\big\}\\
&= \Delta_X \cup \big\{(x,\,r\,x) : \quad x\in\{0\}^c,\;\; r>0\,\big\}\\
&= \Delta_{\{0\}} \cup \big\{(x,\,r\,x) : \quad x\in\{0\}^c,\;\; r>0\,\big\} = E_\varphi.
\end{aligned}
$$

Namely, for some $x\in X$, we have $\varphi(x)\neq 0 \iff x\neq 0 \iff x\in\{0\}^c$. Moreover, for some $x\in\{0\}^c$ and $y\in X$, we have $(x,\,y)\in E_\varphi$ if and only if $y=r\,x$ for some $r>0$.

Now, if $x,\,y\in X$, then by the above observations we can see that, according to Notation 12.6, we have $d_{\varphi\Gamma}(x,\,y)$

$$
= \begin{cases}
0 & \text{if} \quad x=0\,,\quad y=0\,,\\
|\,x-y\,| & \text{if} \quad x\neq 0\,,\quad y/x>0\,,\\
\big|\,|\,x\,|-1\,\big|+1 & \text{if} \quad x\neq 0\,,\quad y=0\,,\\
\big|\,|\,y\,|-1\,\big|+1 & \text{if} \quad x=0\,,\quad y\neq 0\,,\\
\big|\,|\,x\,|-1\,\big| + \frac{\big|\,x\,|\,y\,|-|\,x\,|\,y\,\big|}{|\,x\,|\,|\,y\,|} + \big|\,|\,y\,|-1\,\big| & \text{if} \quad x\,y\neq 0\,,\quad y/x\not>0\,.
\end{cases}
$$

**Remark 15.4** Note that if $\Gamma$ is as in Example 9.9, then by Example 11.6 we also have $Q_{\varphi\Gamma}=E_\varphi$. Therefore, $d_{\varphi\Gamma}$ is again as above. The case when $\Gamma$ is as in Example 10.1 or 10.3 is more difficult.

**Remark 15.5** In addition to Examples 13.1, 14.1 and 15.3, it would be useful to consider the case when $\varphi$ is the natural projection of the unit ball or square in $X$ and $\Gamma$ is one of the relations given in Examples 9.9, 9.10, 10.1 and 10.3.

Moreover, it would also be useful to find some further collinearity or pre-collinearity relations for $X$. And to establish some additional axioms to the ones given in Section 9 in order that we could get a relational characterization of the natural collinearity relation given in Example 10.3.

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