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α-Theory of Hald's method*

ABSTRACT. In this paper, we present new results concerning α-theory for the Hald method to approximate a zero of an analytic function defined on Banach spaces. Hald's method is convenient since there is a quadratic convergence towards the zero and the calculation of the inverse of the operator is not required. We also present a numerical experiment to illustrate our results to solve a classical Fredholm integral equation.

KEY WORDS. α-theory, Hald's method, Newton's method, Ulm's method.

1 Introduction and main Results

We consider F an analytic function defined from an open ball $B(x_0, r) \subset \mathcal{E}$ into $\mathcal{F}$ such that $DF(x_0)^{-1}$ exists where $\mathcal{E}$ and $\mathcal{F}$ are two Banach spaces. We note $L(\mathcal{F}, \mathcal{E})$ the set of linear maps from $\mathcal{F}$ to $\mathcal{E}$. We denote indifferently by $|| \, ||$ a norm in $\mathcal{E}$ or $\mathcal{F}$. Throughout the text ζ is a zero of F. Newton's method is certainly the most well known to solve non linear equations $F(x) = 0$. It consists in defining the sequence from x_0 by

$$x_{k+1} = x_k - DF(x_k)^{-1}F(x_k), \quad k \geq 0. \tag{1.1}$$

A quick search on Mathscinet with keywords Newton's method gives almost 9000 answers. Less known is Hald's method [7] defined by the sequence

$$B_0 \in L(\mathcal{F}, \mathcal{E}) \tag{1.2}$$

$$x_{k+1} = x_k - B_kF(x_k), \quad k \geq 0 \tag{1.3}$$

$$B_{k+1} = 2B_k - B_kDF(x_{k+1})B_k, \quad k \geq 0. \tag{1.4}$$

Here we will treat the case where B_0 is an approximation of $DF(x_0)^{-1}$. Note that (1.4) is one step of Newton's method applied to the equation $G(B) = B^{-1} - DF(x_{k+1}) = 0$. We see

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that Hald's method is "inversion free" and can be interesting when the derivative operator of F is difficult to compute. Note that under the condition $||BDF(x_{k+1}) - I|| < 1$ we can prove easily the quadratic convergence of the sequence (1.4) from $B_0 = B$ which is based on the following equality :

$$I - B_{k+1}DF(x_{k+1}) = (I - B_kDF(x_{k+1}))^2. \tag{1.5}$$

This method to approximate the inverse is known as Schulz method [13].
The goal of this paper is to do an α-theory of the Hald's method. The α-theory consists in giving conditions obtained from a point which imply the convergence of Hald sequence towards a solution of $F(x) = 0$. In the classical Kantorovich's theory the conditions of convergence are obtained from a behaviour of the second derivative of F in a ball as it is the case for a function of class C^2. This reduction of punctual conditions is possible only if the function F is analytic. These punctual conditions are established thanks to following quantities :

$$\begin{aligned}
\beta(F,B,x) &= ||BF(x)|| \\
\gamma(F,B,x) &= \max_{j\geq 2}\left(\frac{1}{j!}||BDF^{(j)}(x)||\right)^{1/(j-1)} \\
\alpha(F,B,x) &= \beta(F,B,x)\gamma(F,B,x) \\
\delta(F,B,x) &= ||I - BDF(x)||.
\end{aligned}$$

We will denote β_k, γ_k, α_k and δ_k respectively for $\beta(F,B_k,x_k)$ etc... We also denote $\beta = \beta_0$, $\gamma = \gamma_0$, $\alpha = \alpha_0$ and $\delta = \delta_0$. Finally we introduce the dominating function

$$h(\tau) = \alpha - (1-\delta)\tau + \frac{\tau^2}{1-\tau}.$$

and the associated Hald sequence defined by

$$\begin{aligned}
b_0 &= -1,\ \tau_0 = 0 \\
\tau_{k+1} &= \tau_k - b_kh(\tau_k), \quad k \geq 0. \\
b_{k+1} &= (2 - b_kh'(\tau_{k+1}))b_k.
\end{aligned} \tag{1.6}$$

We will also use in the sequel the following quantities :

$$\delta_k = 1 - b_kh'(\tau_k), \quad k \geq 0 \tag{1.7}$$

We also denote by

$$\psi(\tau) = (-\delta + 2)\,\tau^2 + (2\delta - 4)\,\tau + 1 - \delta \tag{1.8}$$

the function such that $h'(\tau) = -\dfrac{\psi(\tau)}{(1-\tau)^2}$ and by

$$d = \alpha^2 + 2(\delta - 3)\alpha + (1-\delta)^2 \tag{1.9}$$

the discriminant of the polynomial $(1-\tau)h(\tau)$.

The first result gives a punctual condition of existence of a zero of F. It is a quantitative version of the classical Rouché's theorem using the fact that B_0 approximates $DF(x_0)^{-1}$ and α is less than a certain constant. This result does not depend on any numerical method. Moreover it brings out the dominating function $h(t)$ which plays a central role in the analysis of a Hald sequence.

Theorem 1.1 *Let $0 \le \delta < 1$ and $\alpha < 3 - \delta - 2\sqrt{2-\delta}$. Then the strictly convex function $h(\tau)$ has two real roots $0 < \tau_- < \tau_+ < 1$. Suppose that F is analytic in the ball $B\left(x_0, \dfrac{1-\delta}{\gamma(2-\delta)}\right)$ and consider r such that $\tau_- < \gamma r < \tau_+$. Then F has an unique zero $\zeta \in B(x_0, r)$.*

This result generalizes the result given in Proposition 1.11 of [3] that treats the case $B_0 = DF(x_0)^{-1}$: the condition is $\alpha < 3 - 2\sqrt{2}$. We also remark that the use of Rouché's theorem permits to get the same result as in [15] without the study of the Newton sequence.
The second result gives a punctual sufficient condition for the Hald sequence to converge from x_0. It is an α-theorem.

Theorem 1.2 *For each $0 \le \delta < 1$ there exists $\bar{\alpha}_\delta$ and $q \in [0, 1[$ such that for all x_0 so that $\alpha \le \bar{\alpha}_\delta$ we have $\alpha < 3 - \delta - 2\sqrt{2-\delta}$ and the Hald sequence $(x_k)_{k\ge 0}$ converges quadratically towards a zero ζ of F. More precisely we have for $k \ge 1$,*

$$\gamma||x_k - \zeta|| \le \tau_- - \tau_k \le (\tau_+ - \alpha)q^{2^{k-1}},$$

and

$$||I - B_k DF(x_k)|| \le q^{2^{k-1}}.$$

Theorem 4.3 specifies the values of $\bar{\alpha}_\delta$ and q.

We now state a γ-theorem relatively to the Hald sequence. This result gives a ball $B(\zeta, r)$ in which the Hald sequence converges quadratically to ζ for any initial point $x_0 \in B(\zeta, r)$ and from an approximation B_0 of $DF(\zeta)^{-1}$.

Theorem 1.3 *Let* $z(s) = \dfrac{4-5s-5s^2+4s^3+3s^4}{(1-s-s^2)^2(1-s)}s$ *and* $\bar{s} = 0.1741...$ *such that* $z(\bar{s}) = 1$. *Let* ζ *a zero of* F *such that* $DF(\zeta)^{-1}$ *exists. Denote* $\gamma_\zeta = \gamma(f, DF(\zeta), \zeta)$ *and* $\bar{r} = 1 - \dfrac{\sqrt{2}}{2}$ *and suppose that* F *is analytic in* $B(\zeta, \bar{r})$. *For all* $x_0 \in B\left(\zeta, \dfrac{\bar{r}}{\gamma_\zeta}\right)$ *such that*

$$s = a + e := ||B_0 DF(\zeta) - I|| + \gamma_\zeta ||x_0 - \zeta|| < \bar{s} \tag{1.10}$$

then the map $DF(x_0)^{-1}$ *exists and the Hald sequence converges quadratically to* ζ. *More precisely*

$$e_k := \gamma_\zeta ||x_k - \zeta|| \le z(s)^{2^k-1}s, \quad \textit{and}, \quad a_k := ||B_k DF(\zeta) - I|| \le z(s)^{2^k-1}s.$$

Moreover, we have

$$\delta_k := ||B_k DF(x_k) - I|| \le 3z(s)^{2^k-1}s.$$

Remark 1 It is well known that the quantity $\gamma(F, DF(z), z)^{-1}$ is a lower bound for the radius of convergence of the Taylor series of F in z, see for instance Proposition 6 page 167 of [1]. In the general case it is not easy to estimate the quantity γ. Fortunately this can be done for the polynomial systems of equations or for instance for Fredholm integral operator as in section 9.

Finally we state how Hald's method is related to Newton's method. More precisely we give a condition so that the iterates of Hald sequence and Newton's sequence are close.

Theorem 1.4 *Let* x_k *(respectively* τ_k*) the* k*-th term of Hald sequence associated to* F *(respectively associated to* h*). We also denote by* $\bar{x}_k$ *the Newton iterates starting from* x_k. *The condition*

$$\frac{(1-\tau_k)h(\tau_k)}{\psi(\tau_k)^2} < 3 - 2\sqrt{2}$$

implies that the Newton's sequence defined from x_k *converges quadratically towards a root* ζ *of* F. *We also have :*

$$\gamma||x_{k+1} - \bar{x}_{k+1}|| \le -\frac{h(\tau_k)}{h'(\tau_k)}.$$

In the sequel of paper we give in section 2 a short history and some fundamental works in our context. The proof of Theorem 1.1 is done in section 3. Next, in section 4 we study the behavior of Hald sequences associated to the dominating function $h(\tau)$. Theorems 1.2, 1.3 and 1.4 are proved in sections 5, 6 and 7 respectively. In section 8 we study the case of analytic functions such that $D^jF(x) \equiv 0$ for $j > 2$. Finally, section 9 is devoted to a numerical experiment with a classical Fredholm integral equation. We end with a concluding remark by proposing a study that will extend our result.

2 Short overview and discussion

The Hald sequence is a modified version of Moser's method [11] where $2B_k - B_k DF(x_{k+1})B_k$ replaces $2B_k - B_k DF(x_k)B_k$. This difference implies an order of convergence equal to $(1+\sqrt{5})/2$ for Moser's method. It seems that Moser rediscovered this method first proposed by Ulm [14] who published his article in Russian. The equations $F(x) = 0$ considered comes from the study of stability of solutions of differential equations and/or partial differential equations. More recently in 2008, Ezquerro and Hernández [2] study Hald's method under mild differentiability conditions where there are more precisions and references to the non analytic case. The Kantorovich's classical theory [8] is the beginning of modern studies. Smale [12], drawing inspiration from this theory, had the idea that the complexity of algorithms for solving analytic equations could be expressed only from information given by the function and its derivatives at a point. Then the natural setting of α-theory is that of analytic functions in the ball of convergence of their Taylor series at a point [1]. In this way a criterion of convergence of Newton's method is given by $\alpha(F, DF(x_0)^{-1}, x_0) \leq 3 - 2\sqrt{2}$: this result of semi-local behaviour of Newton's method is named α-theorem. The local behaviour of Newton's method is described thanks the quantity $\gamma(F, DF(\zeta)^{-1}, \zeta)$. More precisely, the convergence of Newton sequence is quadratic for all initial point $x_0 \in B\left(\zeta, \dfrac{5-\sqrt{17}}{2\gamma(F, DF(\zeta)^{-1}, \zeta)}\right)$: this result is named γ-theorem. It is the key point in the measure of complexity to find an initial point for Newton's method using an homotopy method [10].
To our knowledge, third papers have studied the analytic case with the goal to apply it to Fredholm integral equation using Hald's method. The first study [4] constructs a system of recurrence relations in order to prove the convergence of the method. The second study [6] uses the technique of dominating functions becoming from [16] but no result of quadratic convergence are presented. Note that the dominating function is of the type $h(\tau) = \alpha - (1-\delta)\tau + \sum_{j\geq 2} \gamma_j \tau^j$ which well adapted to treat particular case but no universal constant to decide convergence is given. The third study [5] gives a γ-Theorem with the condition $d + 9e < 1/2$ which differs from (1.10). This condition links B_0 with $DF(x_0)^{-1}$ which is not the case in (1.10).

3 Proof of theorem 1.1

To prove the existence of a root of F we use Rouché's theorem. Let us consider the Taylor series of $B_0F(x)$ in x_0. We have $B_0F(x) = B_0F(x_0) + B_0G(x)$ with

$$G(x) = \sum_{k\geq 1} \frac{1}{k!} D^k F(x_0)(x - x_0)^k.$$

Rouché's Theorem states that the analytic functions B_0F and B_0G have the same number of zeros counting the respective multiplicities in the ball $B(x_0, r)$ if for all $x \in \partial B(x_0, r)$ we have $||F(x) - G(x)|| < ||G(x)||$. Since B_0 is invertible the previous assertion holds with the inequality $||B_0(F(x) - G(x))|| < ||B_0G(x)||$.
We first prove that $B_0G(x)$ has only one zero in the ball $B\left(x_0, \frac{1-\delta}{\gamma(2-\delta)}\right)$. Let us consider another zero $y \neq x_0$ of B_0G. Since

$$\begin{aligned} ||B_0DF(x_0)(y - x_0)|| &\geq ||y - x_0|| - ||I - B_0DF(x_0)||\, ||y - x_0|| \\ &\geq (1-\delta)||y - x_0|| \end{aligned} \tag{3.1}$$

we can write

$$\begin{aligned} ||B_0G(y)|| = 0 &\geq \left(1 - \delta - \sum_{k\geq 2} \frac{1}{k!}||B_0D^kF(x_0)||\, ||y - x_0||^{k-1}\right) ||y - x_0|| \\ &\geq \left(1 - \delta - \frac{\gamma\, ||y - x_0||}{1 - \gamma\, ||y - x_0||}\right) ||y - x_0|| \\ &\geq \frac{1 - \delta - (2-\delta)\gamma\, ||y - x_0||}{1 - \gamma\, ||y - x_0||} ||y - x_0||. \end{aligned}$$

Hence $y \notin B\left(x_0, \frac{1-\delta}{\gamma(2-\delta)}\right)$ since $\gamma\, ||y - x_0|| \geq \frac{1-\delta}{2-\delta}$.
It is easy to see that this inequality $||B_0(F(x) - G(x))|| < ||B_0G(x)||$ is verified for all $x \in \partial B(x_0, r)$ if we have :

$$e := ||B_0F(x_0)|| - ||B_0DF(x_0)(x - x_0)|| + \sum_{k\geq 2} \frac{1}{k!}||B_0D^kF(x_0)||\, ||x - x_0||^k < 0.$$

Using (3.1) substituting y by x and $||x - x_0|| = r$, we have

$$\begin{aligned} \frac{1}{\gamma}h(\gamma r) &\geq \beta - (1-\delta)r + \sum_{k\geq 2} (\gamma r)^{k-1} r \\ &\geq e. \end{aligned}$$

Hence the condition $h(\gamma r) < 0$ implies $e < 0$. From Lemma 4.1 below the function $h(\tau)$ is strictly convex for $\tau \in [0, 1[$ and has two distinct real roots satisfying $0 < \tau_- < \tau_+ < \frac{1-\delta}{2-\delta}$ under the conditions $0 < \delta < 1$ and $\alpha < 3 - \delta - 2\sqrt{2-\delta}$. The inequalities $\tau_- < \gamma r < \tau_+$ imply $h(\gamma r) < 0$. This proves Theorem 1.1.

4 The behavior of a Hald sequence associated to $h(\tau)$

The condition of existence of zeros of $h(\tau)$ is given by the following lemma :

Lemma 4.1 *Let $0 \le \delta < 1$. Under the condition $\alpha < 3 - \delta - 2\sqrt{2-\delta}$ the strictly convex function $h(\tau)$ has two real roots*

$$\tau_{\pm} = \frac{\alpha + 1 - \delta \pm \sqrt{d}}{2(2-\delta)} \tag{4.1}$$

where d is defined by (1.9) and moreover

$$\tau_- < \tau_+ < \frac{1-\delta}{2-\delta}.$$

Proof. Under the condition $\alpha < 3 - \delta - 2\sqrt{2-\delta}$ the computation of roots of $h(t)$ is straightforward and the strict convexity is easy to show. The proof $\tau_+ < \dfrac{1-\delta}{2-\delta}$ reduces to $1 - \alpha - \delta - \sqrt{d} > 0$. It easy to see that it suffices that $1 - \alpha - \delta > 0$. The condition $\alpha < 3 - \delta - 2\sqrt{2-\delta}$ implies

$$\begin{aligned} 1 - \alpha - \delta &> 2\sqrt{2-\delta} - 2 \\ &> 0, \quad \text{since } 2\sqrt{2-\delta} - 2 > 0 \text{ for } \delta < 1. \end{aligned}$$

We are done. □

Lemma 4.2 *Suppose $0 \le \delta < 1$ and $\alpha < 3 - \delta - 2\sqrt{2-\delta}$. Hald sequence (1.6) associated to $h(\tau)$ converges towards τ_- with the following properties for $k \ge 0$:*

1. $b_k < 0$.
2. $0 < b_k h'(\tau_k) < 1$. *Consequently* $-b_k < -\dfrac{1}{h'(\tau_k)}$.
3. $0 \le \tau_k < \tau_{k+1} < \tau_-$.

Then, the sequence $(\delta_k)_{k \ge 0}$ defined by (1.7) satisfies

$$\delta_{k+1} \le \delta_k < 1.$$

Proof. The proof is inspired by [6]. From Lemma 4.1 $h(\tau)$ has two roots $0 < \tau_- < \tau_+ < 1$. Prove items 1 to 3 inductively. We have $b_0 = -1$, $b_0 h'(0) = 1 - \delta < 1$ and $0 = \tau_0 < \tau_1 = \alpha$. Prove that $\alpha < \tau^-$. We have

$$\tau_1 = \alpha \le -\frac{h(0)}{h'(0)} = \frac{\alpha}{1-\delta} = \bar{\tau}_1.$$

The real $\bar{\tau}_1$ is one step of Newton's Method associated to $h(\tau)$ from 0. In other words, the point $(\bar{\tau}_1, 0)$ is the intersection of the x-axis and of the tangent line at point $(0, \alpha)$ to graph of $h(\tau)$. The strict convexity of $h(\tau)$ ensures $\alpha < \tau_-$.
Proof of items 1 to 3 for $k+1$ if it is true for k. Clearly, since $h(\tau)$ decreases and is strictly

convex on $[0,\tau_-]$ we can see that $\tau_k \le \tau_{k+1}$ implies $1-b_kh'(\tau_{k+1}) > 1-b_kh'(\tau_k) > 0$. We then get

$$b_{k+1} = b_k + (1-b_kh'(\tau_{k+1})b_k < b_k < 0.$$

Since $\tau_{k+1} < \tau_-$ one has $h'(\tau_{k+1}) < 0$ and $b_{k+1}h'(\tau_{k+1}) > 0$.
Moreover $b_{k+1}h'(\tau_{k+1}) - 1 = 2b_kh'(\tau_{k+1}) - b_k^2h'(\tau_{k+1})^2 - 1 = -(1-b_kh'(\tau_{k+1}))^2 < 0$. Item 3 follows from $\tau_{k+2} - \tau_{k+1} = -b_{k+1}h(\tau_{k+1}) > 0$ and also

$$\tau_- - \tau_{k+2} = (1 - b_{k+1}h'(\lambda_{k+1}))(\tau_- - \tau_{k+1}) > 0.$$

for $\lambda_{k+1} \in]\tau_{k+1}, \tau_{k+2}[$. The inequalities $b_{k+1}h'(\lambda_{k+1}) < b_{k+1}h'(\tau_{k+1})$ permit to conclude $\tau_- - \tau_{k+2} > 0$. Then items 1 to 3 hold.
Now since $b_{k+1} < b_k < 0$ and $h'(\tau_{k+1}) < h'(\tau_k) < 0$ it follows that $\delta_{k+1} = 1 - b_{k+1}h'(\tau_{k+1}) < \delta_k = 1 - b_kh'(\tau_k)$.
This lemma is proved. □

Next, we will use the following quantities :

$$q_0 = \frac{\tau_-}{\tau_+} \text{ and } q_1 = \frac{1-\tau_+}{1-\tau_- - \delta q_0}\left(q_0^2 + \frac{1}{1-\tau_+}\delta q_0\right)$$
$$\delta_1 = 1 - b_1h'(\alpha) = \left(\delta + \frac{(2-\alpha)\alpha}{(1-\alpha)^2}\right)^2$$
$$q = \delta_1 + q_1$$
$$\eta = \frac{1-\tau_+}{1-\tau_- - (1-\alpha)\delta_1 q_1} + (\tau_+ - \alpha)^2$$

Theorem 4.3 *For each $\delta < 1$ there exists $\bar\alpha_\delta$ root of $\eta - 1$ such that for all $\alpha \le \bar\alpha_\delta$ we have $\alpha < 3 - \delta - 2\sqrt{2-\delta}$, $q < 1$ and $\eta \le 1$. Moreover for $\alpha \le \bar\alpha_\delta$ we have :*

$$\delta_k := 1 - b_kh'(\tau_k) \le q^{2^{k-1}} \textit{ and } \quad \tau_- - \tau_k \le (\tau_+ - \alpha)q^{2^{k-1}}, \quad k \ge 1. \tag{4.2}$$

We give some values of $\bar\alpha_\delta$:

δ	0	0.2	0.4	0.6	0.8	1
$\bar\alpha_\delta$	0.165	0.106	0.06	0.022	0.004	0

Proof. Writing $h(\tau) = \dfrac{(2-\delta)(\tau-\tau_-)(\tau-\tau_+)}{1-\tau}$ a straightforward calculation shows that

$$\begin{aligned}\tau_- - \tau_{k+1} &= \tau_- - \tau_k + b_kh(\tau_k)\\ &= \tau_- - \tau_k + \frac{h(\tau_k)}{h'(\tau_k)} - (1-b_kh'(t_k))\frac{h(\tau_k)}{h'(\tau_k)}\\ &= \frac{(2-\delta)(1-\tau_+)}{\psi(\tau_k)}(\tau_k - \tau_-)^2 - \delta_k\frac{h(\tau_k)}{h'(\tau_k)}.\end{aligned} \tag{4.3}$$

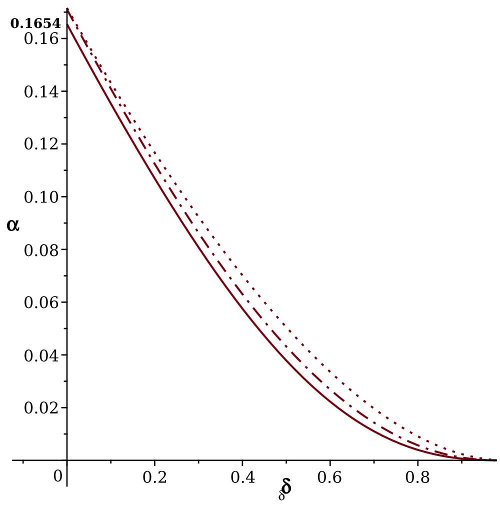

Figure 1: Theorem 4.3.
Curve — : $\eta = 1$.
Curve : −.− : $q = 1$.
Curve : ... : $\alpha = 3-\delta-2\sqrt{2-\delta}$.

In the same way we get :

$$\tau_+ - \tau_{k+1} = \frac{(2-\delta)(1-\tau_-)}{\psi(\tau_k)}(\tau_k - \tau_+)^2 - \delta_k \frac{h(\tau_k)}{h'(\tau_k)}. \tag{4.4}$$

Remember that $h'(\tau) = \dfrac{\psi(\tau)}{(1-\tau)^2} < 0$ for $\tau \in [0, \tau_-]$. Let $q_k = \dfrac{\tau_k - \tau_-}{\tau_k - \tau_+}$, $k \geq 0$. It is easy to see that the function $\tau \in [0, \tau_-] \to \dfrac{\tau - \tau_-}{\tau - \tau_+}$ strictly decreases. From Lemma 4.2 we know $\tau_k < \tau_{k+1}$. Hence $q_{k+1} < q_k$ from which we get the bound

$$\begin{aligned} -\left(\frac{(2-\delta)(1-\tau_-)}{\psi(\tau_k)}(\tau_k - \tau_+)^2\right)^{-1} \delta_k \frac{h(\tau_k)}{h'(\tau_k)} &\leq \frac{\delta_k(\tau_- - \tau_k)(1-\tau_k)}{(\tau_+ - \tau_k)(1-\tau_-)} \\ &\leq \frac{1-\tau_k}{1-\tau_-}\delta_k q_k \end{aligned} \tag{4.5}$$

reduces to the lower bound

$$\tau_+ - \tau_{k+1} \geq \frac{(2-\delta)(1-\tau_-)}{\psi(\tau_k)}(\tau_+ - \tau_k)^2 \left(1 - \frac{1-\tau_k}{1-\tau_-}\delta_k q_k\right). \tag{4.6}$$

Using equality (4.3) and bounds (4.5), (4.6) we get with $q_0 = \dfrac{\tau_-}{\tau_+}$:

$$\begin{aligned} q_1 &\leq \frac{1}{1 - \frac{1}{1-\tau_-}\delta q_0}\left(\frac{1-\tau_+}{1-\tau_-}q_0^2 + \frac{1}{1-\tau_-}\delta q_0\right) \\ &\leq \frac{1-\tau_+}{1-\tau_- - \delta q_0}\left(q_0^2 + \frac{1}{1-\tau_+}\delta q_0\right). \end{aligned} \tag{4.7}$$

and for $k \geq 1$:

$$q_{k+1} \leq \frac{1}{1-\frac{1-\alpha}{1-\tau_-}\delta_k q_k}\left(\frac{1-\tau_+}{1-\tau_-}q_k^2 + \frac{1-\alpha}{1-\tau_-}\delta_k q_k\right), \quad \text{since } \tau_1 = \alpha,$$
$$\leq \frac{1-\tau_+}{1-\tau_- - (1-\alpha)\delta_1 q_1}\left(q_k^2 + \frac{1-\alpha}{1-\tau_+}\delta_k q_k\right), \tag{4.8}$$

since $q_k < q_1$ and from Lemma 4.2 $\delta_k < \delta_1$. On another side

$$\delta_{k+1} := 1 - b_{k+1}h'(\tau_{k+1}) = (1 - b_k h'(\tau_{k+1}))^2.$$

One has $0 < 1-b_k h'(\tau_k) = \delta_k < 1-b_k h'(\tau_{k+1}) = \delta_k - b_k h'(\lambda_k)(\tau_{k+1}-\tau_k)$ where $\lambda_k \in]\tau_k, \tau_{k+1}[$. Hence

$$\delta_{k+1} \leq (\delta_k - b_k h'(\lambda_k)(\tau_{k+1}-\tau_k))^2$$
$$\leq \delta_k^2 + (\tau_{k+1}-\tau_k)^2, \quad \text{since } b_k h'(\lambda_k) > 0.$$

We then get

$$\delta_1 = 1 - b_1 h'(\alpha)$$
$$= \left(\delta + \frac{(2-\alpha)\alpha}{(1-\alpha)^2}\right)^2, \tag{4.9}$$

and for $k \geq 1$

$$\delta_{k+1} \leq \delta_k^2 + (\tau_- - \tau_k)^2$$
$$\leq \delta_k^2 + q_k^2(\tau_+ - \tau_k)^2$$
$$\leq \delta_k^2 + (\tau_+ - \alpha)^2 q_k^2. \tag{4.10}$$

Let $\eta = \dfrac{1-\tau_+}{1-\tau_- - (1-\alpha)\delta_1 q_1} + (\tau_+ - \alpha)^2$, $\mu = \dfrac{1-\alpha}{2(1-\tau_+)}(\eta - (\tau_+ - \alpha)^2)$ and $q = q_1 + \delta_1$. We have for $\alpha \leq \bar{\alpha}_\delta$ root of $\eta - 1$:

$$\delta_{k+1} + q_{k+1} \leq \eta q_k^2 + 2\mu\delta_k q_k + \delta_k^2$$
$$\leq (\delta_k + q_k)^2, \quad \text{since from Lemma 4.4 we have } \mu < \eta \leq 1 \text{ for } \alpha \leq \bar{\alpha}_\delta.$$

By induction we then derive that for $k \geq 1$, $\delta_k + q_k = q^{2^{k-1}}$ using $q < 1$ from Lemma 4.4. We then deduce

$$\delta_k \leq q^{2^{k-1}} \text{ and } \quad \tau_- - \tau_k \leq (\tau_+ - \alpha)q^{2^{k-1}}, \quad k \geq 1.$$

□

Lemma 4.4 *Let q, η be defined as in Theorem 4.3 and*

$$\mu = \frac{1-\alpha}{2(1-\tau_+)}(\eta - (\tau_+ - \alpha)^2).$$

Then we have

1. $\eta > \mu$.
2. *For each $\delta < 1$ there exists $\bar{\alpha}_\delta$ such that $\eta - 1 = 0$. Moreover for all $\alpha \leq \bar{\alpha}_\delta$ we have $\alpha < 3 - \delta - 2\sqrt{2-\delta}$, $q < 1$ and $\eta \leq 1$.*

Proof. Let d defined by (1.9). We know $\tau_+ = \dfrac{\alpha + 1 - \delta + \sqrt{d}}{2(2-\delta)}$ from Lemma 4.1. Then $\eta - \mu > \dfrac{\mu}{1-\alpha}(1 + \alpha - 2\tau_+)$. Since $(1 + \alpha - \alpha\delta)^2 - d = (2-\delta)(4\alpha + \delta - \alpha^2\delta) > 0$ we have $1 + \alpha - 2\tau_+ > 0$. This proves $\eta > \mu$.
To prove item 2 we remark for $\delta \in [0, 1[$ be given the function $\alpha \to \tau_-$ increases and the function $\alpha \to \tau_+$ decreases. From this we can show that the functions $\alpha \to q$ and $\alpha \to \eta$ increase. When $\alpha = 0$ a direct calculation shows that $q - 1 = \delta - 1 < 0$. When $\alpha = 0.999 \times (3 - \delta - 2\sqrt{2-\delta})$ a numerical calculation shows that $q - 1 > 0$: see Figure 2. Hence there exists an implicit function $\alpha_1(\delta)$ such that $q - 1 = 0$.
When $\alpha = 0$ and $\delta \in [0, 1[$ be given we have $\eta - 1 = -\dfrac{1-\delta}{(2-\delta)^2} < 0$. On another side when $\alpha = \alpha_1(\delta)$ a numerical calculation shows that $\eta - 1 > 0$: see Figure 3. Hence there exists an implicit function $\alpha_2(\delta)$ such that $\eta - 1 = 0$. Finally we have $\alpha_2(\delta) \leq \alpha_1(\delta) \leq 3 - \delta - 2\sqrt{2-\delta}$ as it is shown in Figures 1 and 4. Then it is sufficient to consider $\bar{\alpha}_\delta := \alpha_2(\delta)$ and the result follows. □

5 Proof of α-Theorem 1.2

Theorem 1.2 follows from Theorem 5.1 below and Theorem 4.3.

Theorem 5.1 *If $\alpha \leq 3 - \delta - 2\sqrt{2-\delta}$ then the scalar Hald sequence defined by (1.6) dominates the Hald sequence $(x_k)_{k\geq 0}$ defined by (1.2), that is,*

$$\gamma\,||x_{k+1} - x_k|| \leq \tau_{k+1} - \tau_k, \quad k \geq 0.$$

Then the sequence $(x_k)_{k\geq 0}$ converges towards a zero ζ of F with

$$\gamma\,||x_k - \zeta|| \leq \tau_- - \tau_k, \quad k \geq 0.$$

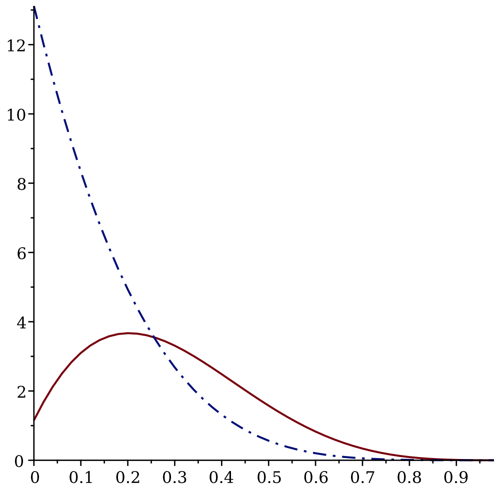

Figure 2: Lemma 4.4.
— : numerator of $q-1$ when $\alpha = 0.999 \times (3-\delta-2\sqrt{2-\delta})$.
–.– : denominator of $q-1$ when $\alpha = 0.999 \times (3-\delta-2\sqrt{2-\delta})$.

Proof. We adapt the proof of Theorem 2 of [6] to our context. We proceed by mathematical induction proving that the following inequalities hold for all $k \geq 0$:

$$\delta_k := ||I - B_k F(x_k)|| \leq 1 - b_k h'(\tau_k) \tag{5.1}$$

$$\gamma ||B_k F(x_k)|| \leq -b_k h(\tau_k) \tag{5.2}$$

$$||B_k D^{(j)} F(x_k)|| \leq -b_k \gamma^{j-1} h^{(j)}(\tau_k). \tag{5.3}$$

For $k = 0$ we have simultaneously $\delta = 1 + h'(0)$, $\gamma||x_1 - x_0|| = \alpha$ and $\gamma(F, B_0, x_0) \leq -\gamma b_0 \gamma(h, b_0, 0) = \gamma$. In fact for $j \geq 2$ we have $h^{(j)}(\tau) = \dfrac{j!}{(1-\tau)^{j+1}}$ and $|h^{(j)}(0)| = j!$.
Let us assume $\gamma\, ||x_{j+1} - x_j|| \leq \tau_{j+1} - \tau_j$ and that these inequalities hold for $j \leq k$. We prove they hold for $k+1$.
We know that

$$I - B_{k+1} DF(x_{k+1}) = (I - B_k DF(x_{k+1}))^2$$
$$I - B_k DF(x_{k+1}) = I - B_k DF(x_k) - \sum_{j \geq 2} \frac{j}{j!} B_k D^{(j)} F(x_k)(x_{k+1} - x_k)^{j-1}$$

It follows

$$\begin{aligned} ||I - B_k DF(x_{k+1})|| &\leq 1 - b_k h'(\tau_k) - \sum_{j\geq 2} \frac{j}{j!} b_k h^{(j)}(\tau_k)(\gamma ||x_{k+1} - x_k||)^{j-1} \\ &\leq 1 - b_k h'(\tau_{k+1}). \end{aligned} \tag{5.4}$$

Hence the inequality (5.1) holds since

$$||I - B_{k+1} DF(x_{k+1})|| \leq (1 - b_k h'(\tau_{k+1}))^2 = 1 - b_{k+1} h'(\tau_{k+1}).$$

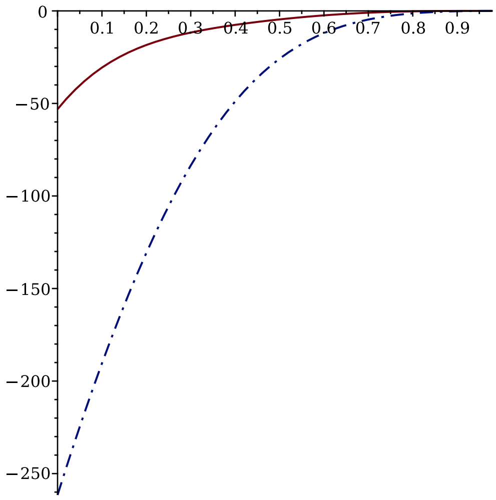

Figure 3: Lemma 4.4.
— : numerator of $\eta - 1$ when $\alpha = \alpha_1(\delta)$
—.— : denominator of $\eta - 1$ when $\alpha = \alpha_1(\delta)$.

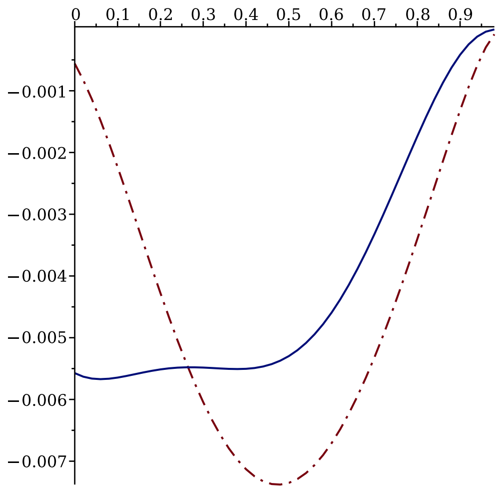

Figure 4: Lemma 4.4.
— : $\bar{\alpha}_\delta - \alpha_1(\delta)$.
—.— : $\alpha_1(\delta) - (3 - \delta - 2\sqrt{2 - \delta})$.

Next, since $B_kF(x_k) + B_kDF(x_k)(x_{k+1} - x_k) = -(I - B_kDF(x_k))(x_{k+1} - x_k)$ we get

$$\gamma B_kF(x_{k+1}) = (I - B_kDF(x_k))\gamma B_kF(x_k) + \sum_{j\geq 2}\frac{1}{j!}B_kD^{(j)}F(x_k)\gamma(x_{k+1} - x_k)^j.$$

We deduce

$$\begin{aligned}\gamma||B_kF(x_{k+1})|| &\leq -(1 - b_kh'(\tau_k))b_kh(\tau_k) - \sum_{j\geq 2}\frac{1}{j!}b_kh^{(j)}(\tau_k)(\tau_{k+1} - \tau_k)^j \\ &\leq -b_kh(\tau_{k+1}).\end{aligned}$$

Then using $B_{k+1}F(x_{k+1}) = (2I - B_kF(x_{k+1}))\, B_kF(x_{k+1})$ the inequality (5.2) follows from

$$\begin{aligned}\gamma||B_{k+1}F(x_{k+1})|| &\leq -(2 - b_kh'(\tau_{k+1}))b_kh(\tau_{k+1}) \\ &\leq -b_{k+1}h(\tau_{k+1}).\end{aligned}$$

From this, it follows $\gamma||x_{k+2} - x_{k+1}|| \leq \tau_{k+2} - \tau_{k+1}$.
Thirdly, inequality (5.3) is satisfied since for $j \geq 2$ we have

$$\begin{aligned}||B_{k+1}D^{(j)}F(x_{k+1})|| &\leq (2 - b_kh'(\tau_k))\sum_{l\geq 0}\frac{1}{l!}(-b_kh^{(l+j)}(\tau_k))\gamma^{j-1}(\tau_{k+1} - \tau_k)^l \\ &\leq -\gamma^{j-1}(2 - b_kh'(\tau_k))b_kh^{(j)}(\tau_{k+1}) = -\gamma^{j-1}b_{k+1}h^{(j)}(\tau_{k+1}).\end{aligned}$$

From Lemma 4.2 the scalar Hald sequence strictly increases and converges to τ_-. This implies the convergence of the sequence $(x_k)_{k\geq 0}$. Theorem 5.1 follows easily. □

6 Proof of γ-theorem 1.3

Here we have $F(\zeta) = 0$ and $\gamma_\zeta = \gamma(F, DF(\zeta)^{-1}, \zeta)$. Let $s_k = e_k + a_k$ where $e_k = \gamma_\zeta||x_k - \zeta||$ and $a_k = ||B_kDF(\zeta) - I||$. We denote $e = e_0$, $a = a_0$ and $s = s_0$.
To prove the convergence of Hald sequence from x_0, we proceed by mathematical induction assuming $s_k \leq z(s)^{2^k-1}s$ for a given index k. For $k = 0$, evidently $s_0 \leq s$, but we need to prove that $DF(x_0)$ is an invertible map. Lemma 2 (b) page 156 of [1] states the map $DF(x_0)$ is invertible for all $x_0 \in B\left(\zeta, \frac{1}{\gamma_\zeta}\bar{r}\right)$ with $\bar{r} = (1 - \sqrt{2}/2)$. Consequently there exists linear map B_0 which approximates $DF(x_0)^{-1}$. For instance $B_0 = DF(x_0)^{-1} + \lambda I$ with $||\lambda DF(x_0)|| < 1$ satisfies $||B_0DF(x_0) - I|| < 1$.

We then write for $k \geq 0$:

$$\begin{aligned}
e_{k+1} &= \gamma_\zeta ||x_k - \zeta - B_k F(x_k)|| \\
&\leq ||I - B_k DF(\zeta) - B_k DF(\zeta) \sum_{j\geq 2} \frac{1}{j!} DF(\zeta)^{-1} D^j F(\zeta)(x_k - \zeta)^{j-1}|| e_k \\
&\leq \left(||I - B_k DF(\zeta)|| + ||B_k DF(\zeta)|| \sum_{j\geq 2} e_k^{j-1} \right) e_k \\
&\leq \left(a_k + (1 + a_k) \frac{e_k}{1 - e_k} \right) e_k \\
&\leq \frac{(a_k + e_k) e_k}{1 - e_k} \\
&\leq \frac{s_k^2}{1 - s_k}. \qquad (6.1)
\end{aligned}$$

On another side a straightforward calculation gives

$$\begin{aligned}
B_{k+1} DF(\zeta) - I &= B_k DF(\zeta) - I + (I - B_k DF(x_{k+1})) B_k DF(\zeta) \\
&= -(B_k DF(\zeta) - I)^2 - \sum_{j\geq 2} \frac{j}{j!} B_k D^j F(\zeta)(x_{k+1} - \zeta)^{j-1} B_k DF(\zeta) \\
&= -(B_k DF(\zeta) - I)^2 - E
\end{aligned}$$

with $E = -B_k DF(\zeta) \sum_{j\geq 2} \frac{j}{j!} DF(\zeta)^{-1} D^j F(\zeta)(x_{k+1} - \zeta)^{j-1} B_k DF(\zeta)$.

We deduce using (6.1) :

$$\begin{aligned}
a_{k+1} &\leq a_k^2 + (1 + a_k)^2 \frac{(2 - e_{k+1}) e_{k+1}}{(1 - e_{k+1})^2} \\
&\leq \frac{3 - 4 s_k^2 - 2 s_k^3}{(1 - s_k - s_k^2)^2} s_k^2. \qquad (6.2)
\end{aligned}$$

From (6.1) and (6.2), a straightforward calculation shows that

$$s_{k+1} \leq \frac{4 - 5 s_k - 5 s_k^2 + 4 s_k^3 + 3 s_k^4}{(1 - s_k - s_k^2)^2 (1 - s_k)} s_k^2 := z(s_k) s_k.$$

Under the condition $s < \bar{s} = 0.1741...$ which implies $z(s) < 1$, we easily deduce with $s_k \leq z(s)^{2^k - 1} s$ that $s_{k+1} \leq z(s)^{2^{k+1} - 1} s$.

On another side we have from the fundamental property (1.5) of $B_k' s$

$$B_{k+1} DF(x_{k+1}) - I = -(B_k DF(x_{k+1}) - I)^2. \qquad (6.3)$$

Moreover

$$B_k DF(x_{k+1}) - I = B_k DF(\zeta) - I + B_k DF(\zeta) \sum_{j\geq 2} \frac{j}{j!} DF(\zeta)^{-1} D^j F(\zeta)(x_{k+1} - \zeta)^{j-1}. \qquad (6.4)$$

Combining (6.3) and (6.4) we get

$$d_{k+1} \leq \left(a_k + (1+a_k) \frac{(2-e_{k+1})e_{k+1}}{(1-e_{k+1})^2} \right)^2. \tag{6.5}$$

Using $a_k, e_k \leq z(s)^{2^k-1}s$, it follows

$$\begin{aligned} d_{k+1} &\leq \left(1 + (1+s) \frac{(2-z(s)s)z(s)}{(1-z(s)s)^2} \right)^2 z(s)^{2^{k+1}-2} s^2 \\ &\leq \left(1 + (1+\bar{s}) \frac{2-\bar{s}}{(1-\bar{s})^2} \right)^2 \bar{s}\, z(s)^{2^{k+1}-1} s \\ &\leq 3 z(s)^{2^{k+1}-1} s, \end{aligned}$$

since the function $s \in [0, \bar{s}[\rightarrow \left(1 + (1+s) \frac{(2-z(s)s)z(s)}{(1-z(s)s)^2} \right)^2 z(s)^{-1} s$ increases. Then the theorem holds.

7 Proof of theorem 1.4

We use the following α-theorem [15] which established the Newton sequence is well defined and converges quadratically from an initial point x_0 provided that $\alpha(F, DF(x_0), x_0) < 3 - 2\sqrt{2}$. We consider the $(k+1)$-th step in the iteration. We remark

$$x_{k+1} - ((B_k DF(x_k))^{-1} - I) B_k F(x_k) = x_k - DF(x_k)^{-1} F(x_k). \tag{7.1}$$

We denote by $\bar{x}_{k+1}$ the left hand side of (7.1) that is the result of one step of Newton's method. We now bound $\alpha(F, DF(x_k), x_k)$ with respect to $\alpha(F, B_k, x_k)$. We first have

$$\begin{aligned} DF(x_k)^{-1} F(x_k) &= (B_k DF(x_k))^{-1} B_k F(x_k) \qquad (7.2) \\ &= (I - (I - B_k DF(x_k)))^{-1} B_k F(x_k). \qquad (7.3) \end{aligned}$$

Using inequality (5.4) of the proof of Theorem 1.2 we find

$$\beta(F, DF(x_k), x_k) \leq (b_k h'(\tau_k))^{-1} \beta(F, B_k, x_k) \tag{7.4}$$

In the same way we get

$$\begin{aligned} \gamma(F, DF(x_k), x_k) &\leq ||(B_k DF(x_k))^{-1}|| \gamma(F, B_k, x_k) \\ &\leq (b_k h'(\tau_k))^{-1} \gamma(F, B_k, x_k). \qquad (7.5) \end{aligned}$$

Hence

$$\alpha(F, DF(x_k), x_k) \leq (b_k h'(\tau_k))^{-2} \alpha(F, B_k, x_k) \tag{7.6}$$

Always from (5.2) and (5.3) we get

$$\alpha(F, B_k, x_k) \le b_k^2 \frac{h(\tau_k)}{(1-\tau_k)^3}.$$

Hence the condition

$$\left(b_k h'(\tau_k)\right)^{-2} b_k^2 \frac{h(\tau_k)}{(1-\tau_k)^3} = \frac{(1-\tau_k)h(\tau_k)}{\psi(\tau_k)^2} < 3 - 2\sqrt{2} \tag{7.7}$$

implies that the Newton's sequence defined from x_k converges quadratically towards a root ζ of F. From (7.1) we get

$$\gamma ||x_{k+1} - \bar{x}_{k+1}|| \le -\frac{h(\tau_k)}{h'(\tau_k)}. \tag{7.8}$$

We are done.

8 Study of a particular case

We give without proof the particularities of previous results when $D^{(j)}F(x) \equiv 0$ for $j > 2$. In this way the dominating function becomes

$$h(\tau) = \alpha - (1-\delta)\tau + \tau^2.$$

with zeros $\tau_\pm = \dfrac{1}{2}\left(1 - \delta \pm \sqrt{d}\right)$ where $d = \sqrt{(1-\delta)^2 - 4\alpha}$.
We have successively

1. Theorems 1.1 and 1.2 hold with $\delta < 1$ and $\alpha < \dfrac{1}{4}(1-\delta)^2$.
2. Theorem 4.3 holds with :
   (a) $q_0 = \tau_-/\tau_+$ and $q_1 = \dfrac{1-\tau_+}{1-\delta q_0}(q_0^2 + \dfrac{1}{1-\tau_+}\delta q_0)$ and
   $\delta_1 = \left(\delta + \dfrac{(2-\alpha)\alpha}{(1-\alpha)^2}\right)^2.$
   (b) $q = \delta_1 + q_1$
   (c) $\eta = \dfrac{1-\tau_+}{1-\tau_- - (1-\alpha)\delta_1 q_1} + (\tau_+ - \alpha)^2.$
   (d) $\alpha < \frac{1}{4}(1-\delta)^2$.
   (e) $\bar{\alpha}_\delta = \alpha_2(\delta)$ root of $\eta - 1$.
   We also give some values of $\bar{\alpha}_\delta$:

δ	0	0.2	0.4	0.6	0.8	1
$\bar{\alpha}_\delta$	0.241	0.148	0.077	0.03	0.005	0

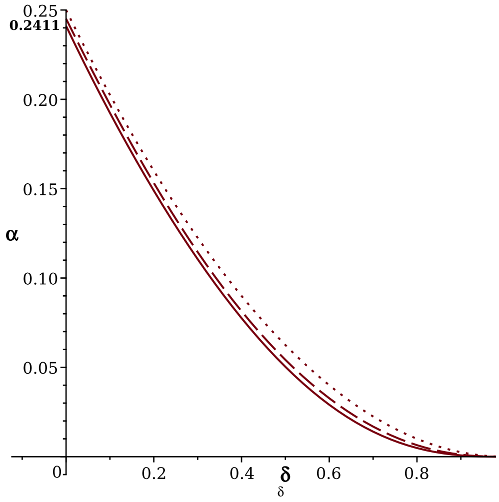

Figure 5: Particular case.
Curve — : $\eta = 1$.
Curve – – : $q = 1$.
Curve ... : $\alpha = 3 - \delta - 2\sqrt{2-\delta}$.

3. Theorem 1.3 holds with $\bar{r} = \frac{1}{2}$, $z(s) = (4s+3)(2s^2+1)s$ and $\bar{s} = 0.23...$ solution of $z(s) - 1 = 0$.
4. Theorem 1.4 holds under the condition $\frac{h(\tau_k)}{h'(\tau_k)^2} < \frac{1}{4}$.

Figure 5 shows respectively the curves $\eta - 1 = 0$, $q - 1 = 0$ and $4\alpha - (1-\delta)^2 = 0$. Figure 6 illustrates numerically that $\bar{\alpha}_\delta = \alpha_2(\delta) < \alpha_1(\delta)) < 3 - \delta - 2\sqrt{2-\delta}$.

9 Numerical experiments

We illustrate the previous results considering the following Fredholm integral operator

$$F(x)(s) = x(s) - 1 - \frac{1}{2}\int_0^1 \sin(st)x(t)^2 dt \tag{9.1}$$

which appears in [9] page 552. We are proving that the following two functions

$$x_{1,0}(s) = 1 \tag{9.2}$$

$$x_{2,0}(s) = 1 + \frac{1}{2}\int_0^1 \sin(st)dt = 1 + \frac{1-\cos(s)}{2s} \tag{9.3}$$

are close to a solution of (9.1). We use the classical max norm in the space of continuous functions. A straightforward calculation shows that with $B_0 = I$

1– $(I - B_0 DF(x))y(s) = \int_0^1 \sin(st)x(t)y(t)dt.$

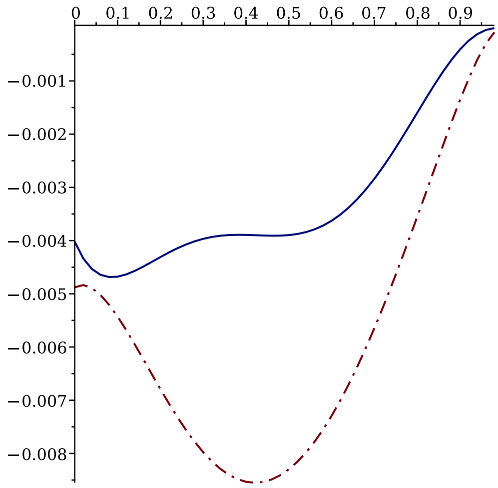

Figure 6: Lemma 4.4.
— : $\alpha_2(\delta) - \alpha_1(\delta)$.
−.− : $\alpha_1(\delta) - (3 - \delta - 2\sqrt{2-\delta})$.

2– $\frac{1}{2}B_0 D^2 F(x) y^2(s) = -\frac{1}{2}\int_0^1 \sin(st) y(t)^2 dt.$

Since the maximum of $\int_0^1 \sin(st)dt = \frac{1-\cos(s)}{s}$ on $[0,1]$ is obtained for $s = 1$ we can numerically compute the quantities β_i, δ_i, γ_i, α_i, q_i, η_i, $\eta_i q_i$ corresponding to $x_{1,0}$ and $x_{2,0}$ respectively. The results are given in Table 1. From Theorem 1.1 we deduce the existence of

	β_i	δ_i	γ_i	$\alpha_i < (1-\delta_i)^2/4?$	η_i	q_i	$\eta_i q_i < 1?$
$x_{1,0}$	0.23	0.46	0.23	$0.05 < 0.07$	1.29	0.49	$0.64 < 1$
$x_{2,0}$	0.08	0.27	0.11	$0.009 < 0.13$	1.51	0.08	$0.11 < 1$

Table 1: Quantities β_i, δ_i, γ_i, α_i, q, η_i, $\eta_i q_i$ relatively to $x_{1,0}$ and $x_{2,0}$.

only one solution in the closed ball $B(x_{1,0}, \tau_{-,i}/\gamma_i)$, $i = 1, 2$ where the $\tau_{-,i}$'s are respectively the first positive root of $h_i(\tau) = \alpha_i - (1-\delta_i)\tau + \tau^2$. Here we have $\tau_{-,1}/\gamma_1 = 0.56$ and $\tau_{-,2}/\gamma_2 = 0.11$.

We perform the computation of Hald sequence in the space of functions defined by

$$\{1 + a_1 \frac{1-\cos(s)}{s} + \sum_{i=1}^{n-1} a_{i+1} s^i \quad \text{where} \quad (a_1, \dots, a_n) \in \mathbb{R}^n\}.$$

We take here $n = 5$. Then the equation $F(x)(s) = 0$ reduces to a polynomial system whose solution approximates that of (9.1). Finally, Hald sequence from $x_{1,0}$ or $x_{2,0}$ associated at this polynomial system converges to the function

$$\zeta(s) = 1 + 0.86 \frac{1-\cos(s)}{s} - 0.04s + 4 \times 10^{-7} s^2 + 9 \times 10^{-4} s^3 + 3 \times 10^{-6} s^4. \tag{9.4}$$

Thanks to this solution we determine the ball of local quadratic convergence given by γ-Theorem 1.3. To do that we compute $\gamma(F, DF(\zeta), \zeta) = \frac{1}{2} ||DF(\zeta)^{-1} D^2F(\zeta)||$.

Proposition 9.1 *We have*

1. $||D^2F(\zeta)|| = || \int_0^1 \sin(st)dt|| < 0.46.$
2. $\gamma_\zeta = \gamma(F, DF(\zeta), \zeta) \le \frac{1}{2} \frac{||D^2F(\zeta)||}{1 - || \int_0^1 \sin(st)\zeta(t)dt||} < 0.54$

For $\bar{s} = 0.23$, *the convergence of Hald's sequence is quadratic for all* $x_0 \in B(\zeta, 0.43)$.

Proof. To determine $DF(\zeta)^{-1}z(t)$ we need to solve $DF(\zeta)y(s) = z(s)$, i.e.

$$y(s) - \int_0^1 \sin(st)\zeta(t)y(t)dt = z(s).$$

Hence if $DF(\zeta)^{-1}$ exists it satisfies

$$z(s) + \int_0^1 \sin(st)\zeta(t)DF(\zeta)^{-1}z(t)dt = DF(\zeta)^{-1}z(s).$$

A numerical computation gives $||I - DF(\zeta)^{-1}|| = || \int_0^1 \sin(st)\zeta(t)dt|| < 0.58\dots$ Hence $DF(\zeta)$ is invertible and

$$||DF(\zeta)^{-1}|| = \frac{1}{1 - || \int_0^1 \sin(st)\zeta(t)dt||} < 2.34$$

Moreover $||D^2F(\zeta)|| = || \int_0^1 \sin(st)|| < 0.46$. Then

$$\gamma(F, DF(\zeta), \zeta) < \frac{1}{2} \frac{||D^2F(\zeta)||}{1 - || \int_0^1 \sin(st)\zeta(t)dt||} < 0.54.$$

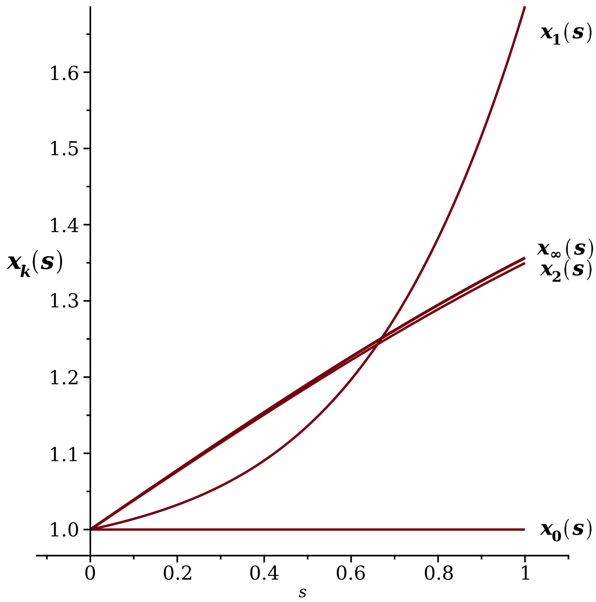

Figure 7: Iterated curves from $x_{1,0}$.

Then, letting $\bar{s} = 0.23$ we get $r = \dfrac{\bar{s}}{\gamma_\zeta} = 0.43$ and from Theorem 1.3 we deduce that the Hald sequence converges quadratically for all $x_0 \in B(\zeta, 0.43)$. □

Table 2 corroborates the quadratic convergence of Hald sequence and Figures 7 and 8 show the iterated curves corresponding to Hald sequence initialized respectively to $x_{1,0}$ and $x_{2,0}$.

k	1	2	3	4	5	6	7
$x_0{=}x_{1,0}, \|\|x_{k+1}-x_k\|\|$	0.22	0.78	0.028	2×10^{-5}	5×10^{-11}	2×10^{-22}	2×10^{-45}
$x_0{=}x_{2,0}, \|\|x_{k+1}-x_k\|\|$	0.08	0.33	2×10^{-3}	2×10^{-7}	5×10^{-15}	2×10^{-30}	3×10^{-61}

Table 2: Quadratic convergence of Hald's sequence.

10 Concluding remark

We first have given a new condition of existence of a zero of an analytic function. Next, we have studied the local behaviour of numerical Hald's method. An application of the results presented here is to find by homotopy method a point satisfying the assumptions of our results. Classically this is realized using Newton's method. It could be interesting to study

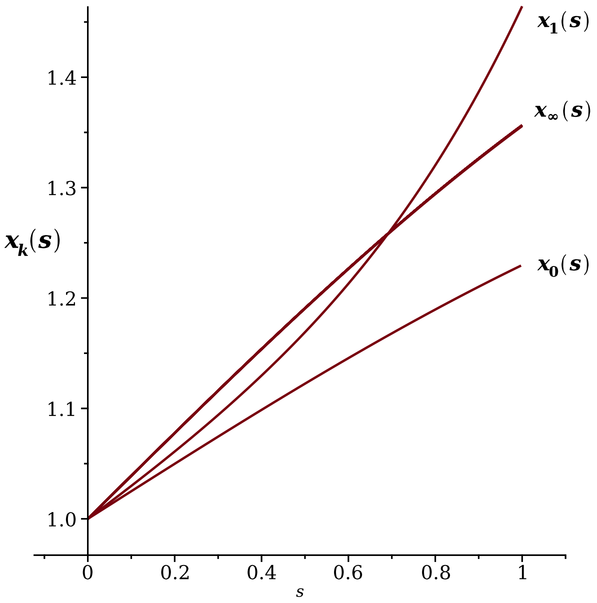

Figure 8: Iterated curves from $x_{2,0}$.

the use of homotopy coupled with Hald's method, for instance in the multiscale methods, to numerically solve integral equations.

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