Synchronisation of non-identical systems by non-invasive mutual time-delayed feedback

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Inspired by time-delayed feedback control it is shown that synchronisation of non-identical systems can be achieved by mutual time-delayed feedback with an asymptotically vanishing interaction. An analytic perturbation scheme is developed which uncovers the merits as well as the constraints of such an approach. As an application the use of the concept for a secure communication channel is considered.

Time-delayed feedback is known for its ability to stabilise periodic motion, that means limit cycles, in dynamical systems. In this short note I will demonstrate that suitable time-delayed feedback is also able to stabilise certain quasiperiodic unstable states, that means torus solutions, in simple oscillator systems.

I. INTRODUCTION

Synchronisation is one of the most prominent concepts in dynamical systems theory with substantial impact for applications. Within dynamical systems theory the study of its various facets has a long-standing tradition[1]. Control problems, a particular type of synchronisation setup with huge relevance in engineering, had their heydays in physics at about a decade ago[2]. Time-delayed feedback autosynchronisation is probably the most interesting among the plethora of schemes which have been rediscovered and developed within the physics community, because of the complex phase space structure introduced by time delay. Hence, it is worth to be studied for its own sake, see e.g.[3] for a contribution from an applied mathematics perspective. Of course, impact of delay on control problems is a well studied topic in the engineering context[4].

The focus of research has shifted to problems including disorder, diversity or more general complex coupled structures, see e.g.[5]. In these studies diversity and disorder is usually considered as a challenge to cope with. Here we follow a different paradigm and consider diversity as a crucial ingredient for synchronisation. The focus will be on the somehow counter-intuitive question of synchronisation of non-identical systems by tiny interaction forces.

To be specific let us consider just two dynamical systems $\dot{z}_t^{(\ell)} = f_\ell(z_t^{(\ell)})$, $\ell = 1,2$ each with an unstable periodic orbit $\zeta_t^{(\ell)} = \zeta_{t-\tau_\ell}^{(\ell)}$ with period τ_ℓ. Inspired by the original idea of Pyragas control[6] we aim at stabilising this orbit using mutual time-delayed feedback

$$\dot{z}_t^{(1/2)} = f_{1/2}(z_t^{(1/2)}) + K_{1/2}(z_t^{(2/1)} - z_{t-\tau_{2/1}}^{(2/1)})\,. \tag{1}$$

It is not at all obvious whether such a coupling is able to stabilise simple dynamical states, here a torus solution. For instance, if one utilises a mutual coupling without time-delayed feedback the stabilisation of time independent states may become impossible as the trace of the Jacobian matrix is not affected by the feedback term. Of course, such a heuristic argument does not preclude anything for the time-delay feedback scheme in eq.(1).

The simple toy model set up in eq.(1) may look as an insult to all those who have investigated synchronisation in realistic scenarios and with far more subtle models. In fact, it is likely that the subsequent considerations are at least indirectly covered by predominantly numerical studies of synchronisation which can be found in the existing literature in abundance. Here we will offer an analytical view on synchronisation at the expense of the complexity of the underlying dynamical model. I that sense the content of this short note may supplement the more realistic studies of real world systems where simple mechanisms are sometimes overshadowed by the required technical complexity of mathematical models.

II. FIXED POINTS: A TRIVIAL FEASIBILITY STUDY

To begin with let us investigate the almost trivial case of stabilising a fixed point. Apart from a proof of concept, the analysis will uncover some features of mutual time-delayed coupling which we will encounter as well in more general situations. For the linear stability problem it is sufficient to restrict from the start to a linear set up $f_\ell(z) = a_\ell z$ with the trivial fixed point at $z_*^{(1)} = z_*^{(2)} = 0$ to be the target. It will turn out to be essential that we allow at least for a two-dimensional phase space for the single unit. Hence, we assume $z^{(\ell)} \in \mathbb{C}$ and $\mathrm{Re}(a_\ell) \geq 0$. The linear system (1) admits exponential solutions $z_t^{(\ell)} = \exp(\Lambda t) u_\ell$ and the stability problem reduces to solve the so called quasi-polynomial following from eq.(1)

$$(\Lambda - a_1)(\Lambda - a_2) = K_1 K_2 (1 - \exp(-\Lambda \tau_1))(1 - \exp(-\Lambda \tau_2)) \,. \tag{2}$$

Clearly stability depends on the product $K_1 K_2$ but not on the individual control gains.

For identical systems, $a = a_1 = a_2$ and $\tau = \tau_1 = \tau_2$ one can take the root of eq.(2), $\Lambda - a = \pm\sqrt{K_1 K_2}(1 - \exp(-\Lambda\tau))$ and eigenvalues can be computed analytically[7]. In particular, a solution with positive real part appears and hence synchronisation of identical systems fails. Thus, diversity of the constituents is a necessity in this setup.

Eq.(2) has infinitely many solutions with negative real part. The actual stability is determined by the few leading roots. Instead of discussing eq.(2) in its full glory consider the marginal case $a_\ell = i\alpha_\ell$ and apply a simple perturbative treatment in terms of the control gain $K_1 K_2$. The two leading roots just follow by solving eq.(2) formally for Λ and using the lowest order result given by $i\alpha_\ell$

$$\Lambda = i\alpha_\ell - K_1 K_2 c_\ell + \mathscr{O}((K_1 K_2)^2) \tag{3}$$

where

$$c_{1/2} = i\frac{(1-\exp(-i\alpha_{1/2}\tau_1))(1-\exp(-i\alpha_{1/2}\tau_2))}{\alpha_{1/2}-\alpha_{2/1}} . \tag{4}$$

Real parts of eigenvalues become negative, i.e., stabilisation occurs for small values of the control gains, if $\mathrm{Re}(K_1K_2c_\ell) < 0$ for $\ell = 1,2$. Such a condition, which determines a sector in the complex plane close to the origin (see figure 1), can easily be guaranteed as long as c_1/c_2 is not negative. Apart from avoiding identical units the simple perturbation argument has used finite frequencies for individual units, i.e., the instabilities bounding the control domain are Hopf bifurcations. In fact, with the choice $\Lambda = i\Omega$ eq.(2) provides a parametric representation of the control boundary in the non-dimensional parameter $w = K_1K_2\tau_1\tau_2$ (see figure 1).

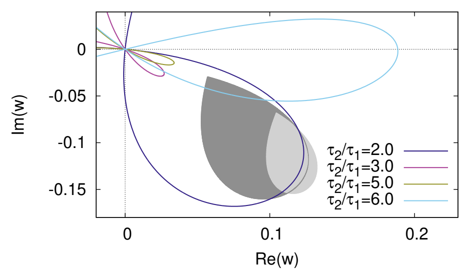

FIG. 1. Bifurcation lines, according to eq.(2) for the critical case $a_1\tau_1 = i$ and $a_2\tau_2 = 4i$ and different values of the ratio τ_2/τ_1. The stable region appears within the loop. A transition at $\tau_2/\tau_1 = 4$ occurs when individual units become identical. Shaded regions are the control domains for $\tau_2/\tau_1 = 2$ and a small finite real part in the parameter, dark: $\mathrm{Re}(a_\ell\tau_\ell) = 0.05$, light: $\mathrm{Re}(a_\ell\tau_\ell) = 0.1$, $\ell = 1,2$.

Results are stable against introducing a small positive real part in a_ℓ, i.e., for control of unstable fixed points. As one would naively expect from related studies, e.g. in simple time-delayed feedback control[8], control domains seem to disappear if the instability of the individual dynamics exceeds a threshold value.

III. STUART-LANDAU OSCILLATORS: AN ANALYTIC CASE STUDY

To gain analytic insight into the scheme defined by eq.(1) we investigate coupled units with an analytically accessible periodic orbit, the famous Stuart-Landau oscillator. Such a paradigmatic model system has been used frequently to explain numerical and experimental results with analytical methods, see e.g.[9]. Here we consider two units defined by

$$f_\ell(z^{(\ell)}) = (\lambda_\ell + i\omega_\ell)z^{(\ell)} + (1+i\gamma_\ell)|z^{(\ell)}|^2 z^{(\ell)} \tag{5}$$

where $z_t^{(\ell)} \in \mathbb{C}$ denotes the complex valued state of the oscillator, ω_ℓ its bare frequency, γ_ℓ the chirp resulting in an amplitude dependent period, and $\lambda_\ell < 0$ the bifurcation parameter in the subthreshold regime. Each unit admits an unstable limit cycle $\zeta_t^{(\ell)} = \sqrt{-\lambda_\ell}\exp(i(2\pi t/\tau_\ell + \phi_\ell))$ with period $\tau_\ell = 2\pi/(\omega_\ell - \gamma_\ell \lambda_\ell)$ and initial phase ϕ_ℓ.

To assess the linear stability of such a solution under non-invasive feedback control, we are going to analyse the corresponding variational equation which written in a rotating frame reads

$$\begin{aligned} \delta\dot{z}_t^{(1/2)} &= -\lambda_{1/2}(1+i\gamma_{1/2})(\delta z_t^{(1/2)} + \delta\bar{z}_t^{(1/2)}) \\ &+ \exp(\mp i(\Delta t + \delta\phi))K_{1/2}(\delta z_t^{(2/1)} - \delta z_{t-\tau_{2/1}}^{(2/1)}) \end{aligned} \tag{6}$$

where $\Delta = 2\pi/\tau_1 - 2\pi/\tau_2$ denotes the detuning of the two periodic solutions, $\delta\phi = \phi_1 - \phi_2$ the initial phase shift, and $\delta z_t^{(\ell)} = (z_t^{(\ell)} - \zeta_t^{(\ell)})\exp(-i(2\pi t/\tau_\ell + \phi_\ell))$ stands for the deviation from the target state. Because of the rotational symmetry of the system, eqs.(1) and (5), the variational equation is periodic in time and constitutes a Floquet problem. Without such a symmetry, i.e., in general, one expects to end up with a quasi-periodic variational equation which is then slightly more elaborate to deal with. As before, we resort to a perturbation expansion, see e.g.[9], with the unfolding parameters λ_ℓ to be considered as the small quantity. At leading order $\mathscr{O}(1)$ eq.(6) can be made autonomous using an appropriate rotating frame. The corresponding quasi-polynomial is in fact given by eq.(2) with $a_\ell = i\omega_\ell$. That, of course, does not come as a surprise since at the lowest order we effectively deal with linear oscillators. The control domain is bounded by Hopf bifurcations and the expression for the bifurcation line is easily derived as

$$w = -\frac{\Omega\tau_1 - 2\pi}{1-\exp(-i\Omega\tau_1)}\frac{\Omega\tau_2 - 2\pi}{1-\exp(-i\Omega\tau_2)} \tag{7}$$

where we have introduced the non-dimensional control gain $w = K_1 K_2 \tau_1 \tau_2$. The imaginary part of the critical eigenvalue $\Lambda = i\Omega$ serves as the parametrisation. The boundary determined by eq.(7) is entirely caused by instabilities of the control loop and does not depend on the internal dynamics of the individual units.

Any constant function is a solution of eq.(6) at order $\mathscr{O}(1)$. Two of these modes are related with the Goldstone modes of the full dynamical system, caused by the translation invariance in both phases on the torus. These critical modes will appear at any order of a perturbation expansion. The two other modes give rise to non-trivial Floquet exponents at order $\mathscr{O}(\lambda)$, i.e., they cause another stability boundary of the control domain. If we introduce $z_t^{(\ell)} = \exp(\Lambda t)u_t^{(\ell)}$,

$\bar{z}_t^{(\ell)} = \exp(\Lambda t)\bar{u}_t^{(\ell)}$ to convert eq.(6) into a Floquet eigenvalue problem, and if we use vector notation $u_t = (u_t^{(1)}, \bar{u}_t^{(1)}, u_t^{(2)}, \bar{u}_t^{(2)})^T$, then the expansion $\Lambda = 0 + \Lambda^{(1)} + \dots$ and $u_t = u^{(0)} + u_t^{(1)} + \dots$ results in

$$\dot{u}_t^{(1)} = B_t(u_t^{(1)} - u_{t-\tau_1}^{(1)}) + C_t(u_t^{(1)} - u_{t-\tau_2}^{(1)}) + f_t \tag{8}$$

with the inhomogeneous part being given by

$$f_t = M_t u^{(0)} = (A^{(1)} - \Lambda^{(1)} - \Lambda^{(1)}\tau_1 B_t - \Lambda^{(1)}\tau_2 C_t)u^{(0)}\,. \tag{9}$$

Here the coefficients read

$$A^{(1)} = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad T_\ell = -\lambda_\ell \begin{pmatrix} 1+\gamma_\ell & 1+\gamma_\ell \\ 1-\gamma_\ell & 1-\gamma_\ell \end{pmatrix} \tag{10}$$

and

$$\begin{aligned} B_t &= \begin{pmatrix} 0 & 0 \\ R_2 & 0 \end{pmatrix}, \quad C_t = \begin{pmatrix} 0 & R_1 \\ 0 & 0 \end{pmatrix} \\ R_{1/2} &= \begin{pmatrix} K_{1/2}e^{\mp i(\Delta t+\delta\phi)} & 0 \\ 0 & \bar{K}_{1/2}e^{\pm i(\Delta t+\delta\phi)} \end{pmatrix}. \end{aligned} \tag{11}$$

The Floquet exponent at first order, $\Lambda^{(1)}$, is determined by the condition that eq.(8) has a periodic solution. It is quite challenging to find elementary and accessible expositions for time dependent cases in the literature[11], even though such secular conditions are well established within the mathematics community[10] and are frequently used in the physics context for time independent setups. The relevant adjoint equation is in fact an advanced delay system

$$\dot{y}_t = -y_t(B_t + C_t) + y_{t+\tau_1}B_{t+\tau_1} + y_{t+\tau_2}C_{t+\tau_2}\,. \tag{12}$$

Its four linearly independent periodic solutions can be computed analytically thanks to the rotational symmetry. They can be conveniently represented as the rows of a fundamental matrix

$$V_t = \begin{pmatrix} 1 & W_1 \\ W_2 & 1 \end{pmatrix} \tag{13}$$

with off diagonal blocks

$$W_{1/2} = \begin{pmatrix} K_{1/2}\theta_{2/1}e^{\mp i(\Delta t+\delta\phi)} & 0 \\ 0 & \bar{K}_{1/2}\bar{\theta}_{2/1}e^{\pm i(\Delta t+\delta\phi)} \end{pmatrix}. \tag{14}$$

Here we have used the abbreviation $\theta_{1/2} = (1-\exp(\pm i\Delta\tau_{1/2}))/(\mp i\Delta)$. The secular condition, i.e. the condition for eq.(8) to admit a periodic solution requires the inhomogeneous part to be "orthogonal" to the solutions of the adjoint equation in the sense that

$$\frac{\Delta}{2\pi}\int_0^{2\pi/\Delta} V_t f_t dt = \frac{\Delta}{2\pi}\int_0^{2\pi/\Delta} V_t M_t dt u^{(0)} = 0\,. \tag{15}$$

Because of the time periodic contributions to the matrices M_t and V_t the off diagonal blocks of the secular matrix vanish and we simply obtain

$$\begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} u^{(0)} = 0 \tag{16}$$

with

$$S_{1/2} = T_{1/2} - \Lambda^{(1)} \begin{pmatrix} 1-w\theta_{2/1} & 0 \\ 0 & 1-\bar{w}\bar{\theta}_{2/1} \end{pmatrix} \tag{17}$$

where again just the non-dimensional combination $w = K_1K_2\tau_1\tau_2$ of control gains enters. In eq.(16) each block S_ℓ gives rise to a vanishing eigenvalue, corresponding to the two Goldstone modes. The two other eigenvalues are real and of order $\mathcal{O}(\lambda)$. The two stability thresholds, caused by transcritical bifurcations, are given by

$$1 = \mathrm{Re}(w(1-i\gamma_{1/2})(1-\exp(\mp i\Delta\tau_{2/1}))/(\pm i\Delta\tau_{2/1}))\,. \tag{18}$$

Eqs.(7) and (18) determine the control domain. The low order results are in quantitative agreement with simulations, see figure 2.

Our perturbation results are not uniformly valid in the detuning Δ. The block diagonal structure in eq.(16) is a consequence of time averaging in eq.(15). For $\Delta = 0$, in particular for identical units, eq.(6) results in an eigenvalue problem which clearly has off diagonal block elements. In line with the analysis of the previous paragraph a straightforward calculation shows that the mutual coupling scheme does not allow for synchronisation of identical units.

IV. AN APPLICATION: A SIMPLE COMMUNICATION PROTOCOL

Within our perturbation scheme synchronisation does not depend on the individual control gains but just on the product w, a property which becomes even rigorous if we use the simpler setup involving linear systems and fixed points as target states. Following a protocol suggested, e.g., in[12] one may implement a secure communication channel by encoding and decoding a message using

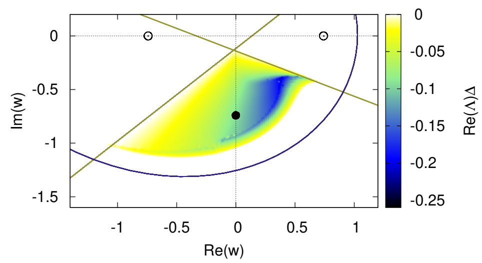

FIG. 2. Control domain for coupled Stuart-Landau oscillators with $\tau_1 = 1$, $\tau_2 = 5/6$, $\gamma_1 = -8$, $\gamma_2 = -12$. Lines: analytic results, eqs.(7) and (18), for the Hopf- (blue) and the transcritical bifurcation (bronze), respectively. Shading indicates the real part of the leading Floquet exponent as obtained from numerical simulations for $\lambda_1 = -0.01$, and $\lambda_2 = -0.02$. Full/open symbols indicate the reference values for successful/unsuccessful secure communication, see figure 3.

the control gains. Suppose the sender and the receiver switch "randomly" between two values $K_1^{\pm}$ and $K_2^{\pm}$ respectively, where $K_1^+ K_2^- = K_1^- K_2^+$ allows for synchronisation and $K_1^{\pm} K_2^{\pm}$ gives an unstable configuration. By monitoring the tiny residual signal that is transmitted through the channel the receiver is able to confirm synchronisation and thus to decode the value set by the sender, while an unstable case is considered to be insecure as an intruder could be able to decode the value set by the sender from the then substantial control signal (see figure 3). Between the transmission of symbols both the sender and the receiver could reset the control gains to a reference value K_ℓ^o with $K_1^o K_2^o = K_1^{\pm} K_2^{\mp}$ to allow for a relaxation of the control signal in case of a previously unsafe transmission. Given the shape of the control domain (see figure 2) we may implement the scheme by a simple modulation of the control phase according to $K_\ell^{\pm} = K_\ell^o \exp(\pm i\pi/4)$.

While at this level there is certainly no proof that the scheme provides secure communication, there is no obvious way to extract the transmitted message from the residual signal as the entire spectrum of the variational equation is not affected by the message. In addition, the size of the eigenmodes does not change either since we just modulate the phase of the control gains. Furthermore, one can still optimise the parameter values, resort to the system with fixed points discussed earlier, or even may use noise in the channel to prevent decryption. Of course, a detailed analysis beyond the simple illustration presented here is certainly required.

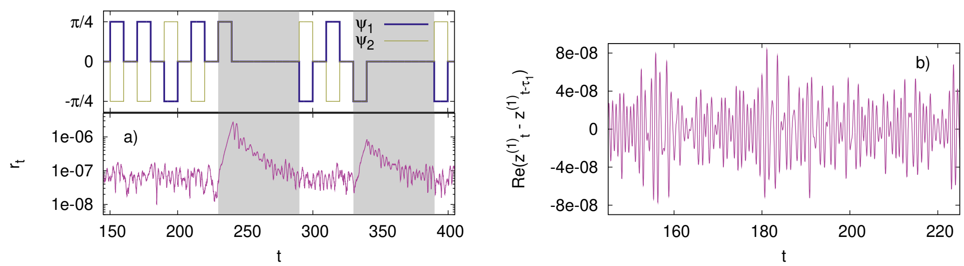

FIG. 3. (a) Piecewise constant control phases (sender: ψ_1, receiver: ψ_2) and the size of the residual signal $r_t = |z_t^{(1)} - z_{t-\tau_1}^{(1)}| + |z_t^{(2)} - z_{t-\tau_2}^{(2)}|$ on a logarithmic scale for a system of coupled Stuart-Landau oscillators with $K_1^o = -K_2^o = 2(1+i)/3$ (other parameters as in figure 2 with $K_\ell = K_\ell^o \exp(i\psi_\ell)$). Periods of signal mismatch, i.e. unsuccessful transmission and subsequent relaxation are shaded. (b) A single component of the residual signal during a successful transmission (see (a)).

V. CONCLUSION

Mutual time-delayed coupling is able to stabilise periodic orbits with different periods, i.e., a torus solution, in a non-invasive way. For the models considered here diversity of the individual units is crucial. In addition, and similar to results in time-delayed feedback control[9,13], a chirp seems to be required as well. The methodology can be generalised to complex coupling schemes of many units, opening the possibility to investigate the impact of distributed delay and of the coupling topology on dynamical features, an aspect which cannot be easily dealt with by conventional master stability function approaches (see e.g.[14] for a contribution in this context).

An experimental implementation of the synchronisation scheme looks promising since Stuart-Landau oscillators have generic properties. Straightforward applications could indeed be in terms of secure communication channels, e.g., in optical networks. Apart from overcoming the crucial technological obstacles one needs a thorough theoretical understanding of the impact of noise and the breaking of the rotational symmetry. The latter turns the stability analysis in a quasi-periodic eigenvalue problem with mode locking features becoming potentially relevant. Apart from these challenges the scheme introduced here ultimately illustrates the naive statement that two way coupling is more than one way coupling.

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AUTHOR DECLARATION

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