Control of Chaos by Time–Delayed Feedback: a Survey of Theoretical and Experimental Aspects

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Abstract. Time–delayed feedback control has been introduced as a powerful tool for control of unstable periodic orbits in dynamical systems. From the experimental point of view its strength is based on the fact that the application of this method requires just the measurement of simple signals, and it has been applied in physics, chemistry and biology. We present an overview of the theoretical foundations of time–delayed feedback methods and explain in detail the implications for real experiments.

1 Introduction

During the last decade control of chaos has developed into one of the most prominent fields in applied nonlinear science [1]. Its beginning was triggered by the observation that on the one hand chaotic motion provides a huge number of unstable states and that on the other hand each of these states can be stabilised by tiny control forces [2].

The control scheme that was originally developed in [2] is based on the local phase space structure in the vicinity of the target orbit. It has been applied in different experimental contexts where such a structure is experimentally accessible (cf. e.g. [3]). In fact, such approaches allow for more sophisticated goals like tracking and targeting of particular trajectories. In the wake of these developments variants of the control method have been rediscovered [4] which are simpler to handle form the experimental point of view and which are often considered in control theory.

Most applications in the physicists context concern the stabilisation of fixed points, either in the original phase space or with respect to a suitable Poincare cross section. Above all it is important to stress that control schemes mentioned above aim at noninvasive control, so that the control force finally vanishes and the target state is not altered by the control loop. Noninvasive schemes open the possibility to use control methods as a kind of nonlinear spectroscopy since different proper eigenmotions of the system can be detected and recorded.

Control theory is a well developed discipline in applied mathematics and engineering science, and there exists a huge amount of literature ranging from elementary to sophisticated presentations (cf. e.g. [5,6]). Control problems are frequently formulated in terms of boundary value and optimisation problems. Thus also global properties like controllability and observability of a target state can be addressed. However, such concepts are to some extent restricted to systems where at least partial information about the internal state is accessible or to systems which are either linear or conjugate to a linear systems in parts of their phase space. Whether a control scheme is invasive does not play a crucial role from the point of view of control theory.

It is a well established fact for decades [7] and to some extent common wisdom in control theory that time delay reduces the efficiency of a control scheme. Therefore it was quite a surprise and has been pointed out recently [8] that time–delayed feedback may be suitable to generate control forces for stabilising time–periodic states. The main idea of such feedback schemes is quite simple and can be applied in different experimental contexts. Suppose a signal $s(t)$ is accessible for measurement and we intend to stabilise an unstable periodic state of period τ. From the time–delayed difference $s(t)-s(t-\tau)$ one generates a control force, e.g. by linear amplification, and one feeds it back to the system. Obviously such a scheme (cf. Fig. 1) is noninvasive since the force vanishes provided the τ–periodic target state is reached. Whether such a prescription works requires a more detailed investigation. Nevertheless, the scheme has been applied successfully in optical experiments, e.g. for stabilising lasers [9] or discharge gas tubes [10], in hydrodynamic experiments e.g. for stabilisation of turbulent Taylor–Couette flows [11], in chemical setups e.g. for electrochemical reactions [12], in magnetic systems, e.g. for controlling high power ferromagnetic resonance experiments [13] and in biological systems, e.g. for controlling arrhythmic cardiac [14]. Of course demonstration experiments like mechanical pendula [15] or electronic circuits [16] have been performed also. The latter are very useful to study details of the control scheme from an experimental perspective.

Despite its experimental success a deeper theoretical understanding of time–delayed feedback control has been gained only recently. Here we present a summary of these developments. We mainly focus on essential theoretical features without giving all the technical details. The relevance of these items is illustrated by results from electronic circuit experiments. Section 2 is devoted to the analysis of the original Pyragas scheme. More specialised topics like multiple delays or the introduction of additional time scales in the control process are addressed in section 3. The lack of a proper theoretical understanding of some aspects is related to the fact that time delay systems act on an infinite–dimensional phase space. Thus, dynamics with time delay is of

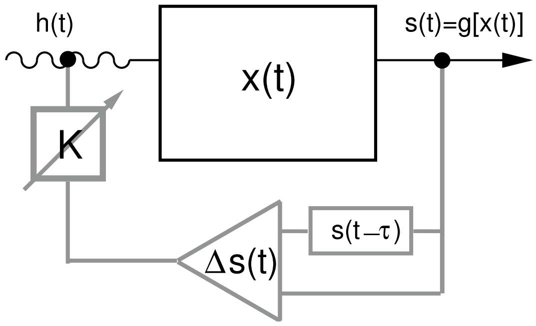

Fig. 1. Diagrammatic view of time–delayed feedback control. Internal degrees of freedom are denoted by $\boldsymbol{x}(t)$. The measured signal $s(t) = g[\boldsymbol{x}(t)]$ depends on the internal degrees of freedom. The control force $F(t) = K[s(t) - s(t-\tau)]$ is generated from the time–delayed difference by linear amplification. The force is coupled to some external parameter $h(t)$ of the experimental setup

interest by itself. We point out some recent trends in the outlook in section 4.

2 Properties of the Original Pyragas Scheme

Theoretical approaches for time–delayed feedback control rely to a good deal on linear stability analysis. Fortunately essential steps can be performed without resort to a particular model. Thus the results and the control performance share a large degree of universality.

Let us just briefly review the main concepts and introduce the essential notations [17]. Let $\boldsymbol{x}(t)$ denote the internal degrees of freedom of the system under consideration. The dynamics without control force is assumed to be governed by a set of equations of motion

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) \quad . \tag{1}$$

We assume that the motion admits an unstable periodic state $\boldsymbol{\xi}(t) = \boldsymbol{\xi}(t+\tau)$ which we finally intend to stabilise. Observing small deviations from the target state $\delta\boldsymbol{x}(t) = \boldsymbol{x}(t) - \boldsymbol{\xi}(t)$ the stability of the orbit is governed by linear analysis. With the usual exponential ansatz $\delta\boldsymbol{x}(t) = \exp(\lambda t)\boldsymbol{q}(t)$, where the eigenmode is periodic in time $\boldsymbol{q}(t) = \boldsymbol{q}(t+\tau)$ by virtue of the periodicity of the unstable orbit, one finally obtains a Floquet eigenvalue problem

$$\lambda_\nu \boldsymbol{q}_\nu(t) + \dot{\boldsymbol{q}}_\nu(t) = D\boldsymbol{f}(\boldsymbol{\xi}(t))\boldsymbol{q}_\nu(t) \quad . \tag{2}$$

$D\boldsymbol{f}$ denotes the Jacobian matrix and the index numbers the different eigenmodes. While the real part of the Floquet exponents λ_ν governs the stability the imaginary part determines the torsion of neighbouring trajectories in phase space. The latter quantities are defined modulo $2\pi/\tau$ owing to the periodicity of the eigenmodes. In what follows we suppose for the simplicity of notation that just the real part of one Floquet exponent is positive. Generalisations of our considerations to cases with several unstable exponents is straightforward to a large extent. We label the unstable exponent by λ_+.

The control system sketched in Fig. 1 contains the control loop which is based on the time–delayed difference. In terms of an equation of motion a particular realisation of the control reads

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) - K\left[\chi \boldsymbol{x}(t) - \chi \boldsymbol{x}(t-\tau)\right] \quad . \tag{3}$$

In eq. (3) the control force has been based on the observation of the full state through the observable $\chi \boldsymbol{x}(t)$, where the factor χ takes care of physical dimensions and can be of course absorbed in the control amplitude. The feedback used in eq. (3) is very special and to some extent not very realistic, since such a diagonal control requires the measurement and the stabilisation of each degree of freedom. But it considerably facilitates a complete analytical discussion. Performing a linear stability analysis and using the exponential ansatz $\delta\boldsymbol{x}(t) = \exp(\Lambda t)\boldsymbol{Q}(t)$ we immediately recognise that the delay term $\delta\boldsymbol{x}(t) - \delta\boldsymbol{x}(t-\tau)$ reduces to a contribution which is local in time, $[1-\exp(-\Lambda\tau)]\exp(\Lambda t)\boldsymbol{Q}(t)$. Thus stability is governed by the ordinary Floquet problem

$$\Lambda_\alpha \boldsymbol{Q}_\alpha(t) + \dot{\boldsymbol{Q}}_\alpha(t) = D\boldsymbol{f}(\boldsymbol{\xi}(t))\boldsymbol{Q}_\alpha(t) - K\chi[1-\exp(-\Lambda_\alpha\tau)]\boldsymbol{Q}_\alpha(t) \quad . \tag{4}$$

The structure of eq. (4) permits to absorb the control term, i.e. the second contribution on the right hand side, in the Floquet exponent, i.e. the first contribution on the left hand side. Then, by comparison with eq. (2) we get the characteristic equations

$$\lambda_\nu = \Lambda_\alpha + K\chi[1-\exp(-\Lambda_\alpha\tau)] \tag{5}$$

where the index α labels the different Floquet exponents of the controlled system. Thus each eigenvalue λ_ν of the uncontrolled system gives rise to a whole family Λ_α of exponents which determine stability in the case with control. Such a structure in fact reflects the phase space with infinite dimension on which the corresponding differential–difference equation (3) operates. Transcendental equations of the kind eq. (5) are typical for the stability analysis of time delay systems [18]. Several techniques, either analytical or numerical, have been developed for their evaluation [19,18,20,21]. In eq. (5) it is often sufficient to consider the unstable free branch, i.e. $\nu = +$.

The Floquet exponents of the controlled system, Λ_α, determine the control performance. Successful control requires $\mathrm{Re}\Lambda_\alpha < 0$. The dependence of the exponents on the control amplitude K is depicted in Fig. 2 for an orbit with $\mathrm{Im}\lambda_+ = \pi/\tau$, i.e. an orbit which flips its neighbourhood during one turn. Such orbits are generated in period doubling bifurcations and therefore occur frequently in chaotic attractors which are generated by period doubling cascades. One obeserves a typical butterfly shaped structure for the leading exponent which results in a finite control interval. At the lower control threshold a flip instability occurs, whereas the upper control threshold is related to a nontrivial imaginary part so that a Hopf instability occurs. All the other complex valued solutions of eq. (5) have lower real part (cf. e.g. [19,22]). If the real part of the free exponent, $\mathrm{Re}\lambda_+$ is increased, then the whole curve shifts essentially upwards. Thus the control interval shrinks and finally vanishes at $\mathrm{Re}\lambda_+\tau = 2$ (cf. Fig. 5) [23]. Thus time–delayed feedback control is limited to low–period or weakly unstable orbits.

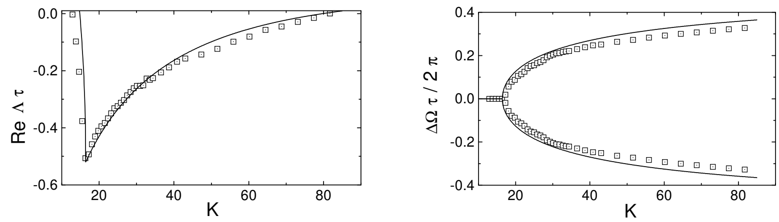

Fig. 2. Leading Floquet exponent of the system subjected to time–delayed feedback control in dependence on the control amplitude. Left: real part, right: imaginary part $\Delta\Omega = \mathrm{Im}(\Lambda - \lambda_+)$. Full line: analytical result according to eq. (5) with $\lambda_+\tau = 1.07 + i\pi$ and $\chi\tau = 0.036$. Symbols: experimental data obtained from a nonlinear diode resonator experiment

So far our analysis was rigorous but restricted to the diagonal control scheme. We compare these results with experimental data obtained in an electronic circuit experiment (cf. for details of the setup [22]). Experimentally the dominant Floquet exponent can be obtained either from observing the transient dynamics or by linear response methods. Although the coupling of control forces is quite different compared to the diagonal scheme eq. (3) we observe a remarkable coincidence when the parameters λ_+ and χ in eq. (5) are considered as fit parameters. Thus the analysis presented above has considerable predictive power.

To understand such a coincidence on a substantial level one may perform a linear stability analysis for a general coupling scheme, i.e. for a measured signal $s(t) = g[\boldsymbol{x}(t)]$ and a coupling of the control force to internal degrees of freedom which is not specified from the beginning [17]. In contrast to eq. (4) there appears now a nontrivial control matrix. Nevertheless the characteristic equation can be cast in the form

$$\Lambda_\alpha = \Gamma_\nu \left[K[1 - \exp(-\Lambda_\alpha \tau)]\right] \qquad (6)$$

where the functions Γ_ν obey the constraints $\Gamma[0] = \lambda_\nu$. They contain all the details of the control scheme. Equation (5) which was used for comparison with the experimental data can be considered as an asymptotic expansion of the full characteristic equation (6). Details of the quality of such an approximation have been discussed in [22]. Above all the considerations make explicit that even the simple expression (5) captures control features beyond the diagonal control.

Equation (6) is hard to evaluate since the function Γ_ν appearing on the right hand side is not known in general. Nevertheless the constraint $\Gamma_+[0] = \lambda_+ \neq 0$ tells us that at the control boundary, $\mathrm{Re}\Lambda_\alpha = 0$, the corresponding imaginary part of the Floquet exponent does not vanishing, since $\Lambda_\alpha = 0$ results in a contradiction. Thus torsion is a necessary ingredient for time–delayed feedback control to work at all [17]. Such a feature can be translated into properties of the free orbit [24]. Only those orbits are accessible to time–delayed feedback schemes which have an odd number of positive unstable Floquet multipliers $\exp(\lambda_\nu \tau)$.

Such a limitation has been derived from the exact characteristic equation (6). One has to modify the feedback scheme in order to remove this constraint, and there are two alternatives available in the literature. On the one hand one may modulate the control amplitude periodically in time, $K = K(t) = K(t + T)$. For modulation periods larger than the period τ of the unstable orbit one may achieve stabilisation even in cases where the plain Pyragas scheme fails [25,26]. Such an idea has been applied successfully in experiments. By such modulations one considerably changes the spectral structure of the associated stability problem (cf. e.g. [27]). In fact, one essentially adds additional degrees of freedom to the control loop. On the other hand a controller containing directly an additional unstable degree of freedom has been proposed recently [28] to avoid the constraint mentioned in the previous paragraph.

We have already stressed that the simple analytical expression (5) captures essential features of time–delayed feedback control beyond the diagonal coupling scheme. Since eq. (5) is in general only an approximation there occur of course deviations when other coupling schemes are considered. Equation (5) fails to be a reasonable approximation of the exact characteristic equation (6) because hybridisation with former stable branches changes the spectral structure considerably. One may employ approximations which go beyond the linear order [22,29] but they become hardly feasible for experimental purpose because of the larger number of free parameters. As an essential new feature the control interval may be limited by Floquet exponents which emerge from formerly stable eigenvalue branches. Figure 3 contains an illustration by experimental data obtained from electronic circuit experiments [22]. One readily sees that the control domain is now severely reduced compared to a prediction based entirely on the simple eq. (5).

The detailed structure of the spectrum is determined by the particular form of the control matrix, i.e. by the details of the measured signal $s(t) = g[\boldsymbol{x}(t)]$ and by the coupling of the control force to the internal degrees of freedom. It is in fact one purpose of control theory to discuss the influence of these features on the control performance for non time–delayed feedback schemes. But almost nothing is known for time–delayed feedback control, mainly because of the transcendental characteristic equation that

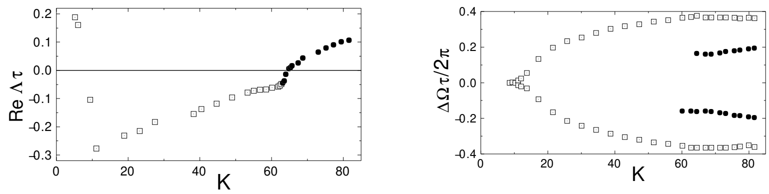

Fig. 3. Floquet branch crossing in an an electronic circuit experiment. Floquet exponents in dependence on the control amplitude, left: real part, right: imaginary part $\Delta\Omega = \mathrm{Im}(\Lambda - \lambda_+)$. Open symbols: branch which determines the lower control threshold, full symbols: nonleading branch which determines the upper control threshold

emerges from the corresponding stability problem. Only preliminary results, i.e. numerical simulations of particular model systems can be found in the literature, e.g. for spatially extended systems [30,31], including global feedback schemes which are particularly easy to implement practically [32]. No consistent picture shows up so that to date no a priori estimate can be given what type of coupling scheme enhances the performance of time–delayed feedback control. Such a problem is one of the challenges for future research.

3 Advanced Problems in Time–Delayed Feedback Control

In the previous section we have addressed several topics concerning the original Pyragas control scheme. However, a lot of problems have not been mentioned so far, and this section is devoted to a discussion of more specialised topics. In particular we will discuss how appropriate delay times τ can be determined, in which way the control performance can be improved by multiple delays, how control loop latency affects the control properties and in which way a broken time translation invariance can contribute to an improvement of the control.

3.1 Adjustment of the Delay Time

To ensure the noninvasive character of time–delayed feedback control the delay time must be adjusted according to the period of the unstable orbit. Only in special cases, e.g. in periodically driven systems such periods are known a priori. There exist however several empirical schemes which can successfully cope with such a problem. It has been already stressed in [8] that some average of the control signal $s(t) - s(t - \tau)$ shows sharp minima when the delay is changed continuously and that these resonance like structures occur at periods of unstable orbits. These features depend on the value of the control amplitude K and it is quite unlikely that a lot of periods can be observed. But one has at least an indicator for the proper adjustment of the delay time. Improvements of this idea can be found in the literature. Instead of observing the control signal other signatures for proper periods, e.g. the distance between local maxima of the signal $s(t)$, can be considered [33] and the whole procedure can be embedded in steepest descent methods [34]. Although in real experiments the control signal never vanishes identically (cf. e.g. [12,13]) one gets at least a first order approximation for a noninvasive scheme.

One often observes periodic signals even if the delay time is not adjusted properly. Such induced periodic behaviour follows from the fact that the system under consideration and the control loop perform a combined dynamics. The period Θ of such an induced periodic behaviour can be related to the period T of proper unstable periodic orbits of the system by employing arguments borrowed from structural stability [18]. An estimate for the induced period Θ in dependence on the true period T and the delay τ can be derived by applying first–order perturbation theory to the full differential–difference equation describing the control loop [35],

$$\Theta = T + \frac{K}{K-\kappa}(\tau - T) + \mathcal{O}((\tau - T)^2) \quad . \tag{7}$$

The result is valid under quite general conditions and the details of the control scheme enter just through the system dependent parameter κ. Whether such an induced periodic behaviour is stable, i.e. whether it shows up in the control signal, depends of course on the value of the control amplitude K. But in certain ranges of control amplitudes the behaviour predicted by eq. (7) can be recorded in real experiments [35] (cf. Fig. 4). Thus a few data points are sufficient to estimate the true period T form the period of the control signal Θ and its dependence on the control amplitude. Of course eq. (7) is a first–order result and its applicability is limited to cases where the difference between delay and true period is not too large.

3.2 Control Schemes with Multiple Delays

As already pointed out in section 2 the original Pyragas scheme is limited to control of unstable orbits with short period. In order to overcome such a limitation control methods have been proposed which use multiple delay times [36]. A quite simple scheme generates control forces from time–delayed differences with an exponentially decaying weight which mimics low pass frequency filters

$$\begin{aligned} F(t) &= K \sum_{\nu=0}^{\infty} R^{\nu} \left[s(t - \nu\tau) - s(t - (\nu+1)\tau) \right] \\ &= K \left[s(t) - s(t-\tau) \right] + R F(t-\tau), \qquad |R| < 1 \quad . \end{aligned} \tag{8}$$

Such a force can be realised without additional delay lines and the scheme is in particular suitable for optical implementation [37].

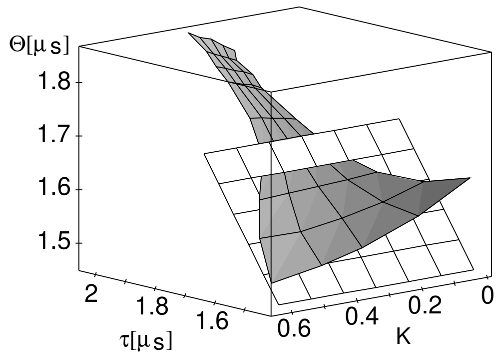

Fig. 4. Period of the control signal in dependence on the control amplitude and the delay (gray surface) obtained from an electronic circuit experiment. Period of the proper orbit $T = 1.656\mu$s. As a guide for the eye the plane $\Theta = \tau$ is displayed also. See [35] for details of the experimental setup

The extended time–delayed feedback method can be discussed analytically within the diagonal coupling scheme. The corresponding characteristic equation results in [23]

$$\lambda_\nu = \Lambda_\alpha + K\chi \frac{1 - \exp(-\Lambda_\alpha \tau)}{1 - R\exp(-\Lambda_\alpha \tau)} \quad . \tag{9}$$

Stability domains for different values of the filter parameters are sketched in Fig. 5. It is quite straightforward to check that stabilisation is possible if $\mathrm{Re}\lambda_\nu \tau < 2(1+R)/(1-R)$ for orbits with $\mathrm{Im}\lambda_+ \tau = \pi$. Thus the introduction of multiple delays increases the control performance.

The previous analysis is of course only approximately valid for general coupling schemes as already discussed in section 2. Nevertheless the main features often endure as can be checked by experiments [23] (cf. Fig. 6). In general control intervals widen if the filter parameter is increased. One should, however, keep in mind that exceptions may occur since nonleading branches in the spectrum may become relevant [22]. Thus the precise shape of the control domain depends on the details of the system and the coupling of the control force as it is for instance visible in diverse numerical simulations [30,31].

The extended control scheme described so far is in principle not limited to filter parameters $|R| < 1$. One may apply an unstable controller with $|R| > 1$ and such a concept, as already stressed in section 2, is useful to overcome the constraint imposed by torsion [28].

The method according to eq. (8) uses a particular version of multiple delays, i.e. integer multiples of the basic delay time. This form aims at a simple experimental realisation. Its theoretical motivation comes from a suitably designed transfer function in order to suppress frequency components which

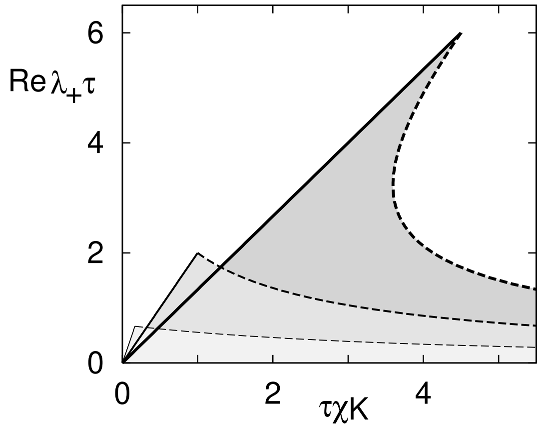

Fig. 5. Control domains in the Reλ–K parameter plane for extended time–delayed feedback control. Analytical results according to eq. (9) for an orbit with Im$\lambda\tau = \pi$ and for different values of the filter parameter: $R = 0.5$ (thick), $R = 0$ (medium), $R = -0.5$ (thin)

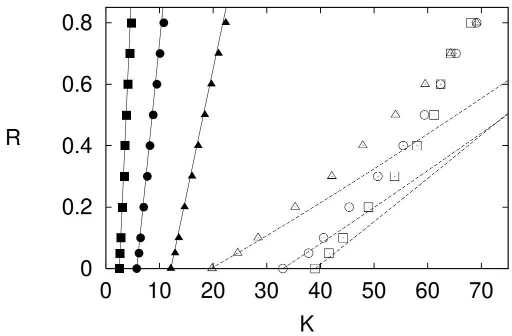

Fig. 6. Control domains in the R–K parameter plane for extended time–delayed feedback control. Experimental results from an electronic circuit experiment for orbits with different Floquet exponents λ (triangles, circles, and squares). Lower control threshold (full symbols), upper control threshold (open symbols), analytical result according to eq. (9) (lines). See [23] for details of the experiment and the chosen parameters

belong to the orbit which will be stabilised. Such a particular choice is not the most general for a time–delayed feedback loop. In fact any control force consisting of terms like $s(t-\delta_i) - s(t-\tau-\delta_i)$ yield a noninvasive control scheme regardless of the values of the offsets δ_i. Whether approaches really increase the control performance is a delicate question. In fact, common wisdom in control theory tells us that additional delays are unlikely to increase the control performance [7]. That is in particular true for control loop latencies which may become important in fast experiments on GHz time scales where no instantaneous feedback can be realised. In terms of the original Pyragas scheme such a loop latency results in

$$F(t) = s(t-\delta) - s(t-\tau-\delta) \quad . \tag{10}$$

The influence of such a single latency δ on time–delayed feedback control has been studied quantitatively in [38]. Following an analysis along the lines of section 2 one obtains for the characteristic equation

$$\lambda_\nu = \Lambda_\alpha + K\chi(\delta)\exp(-\Lambda_\alpha\delta)\left[1-\exp(-\Lambda_\alpha\tau)\right] \quad . \tag{11}$$

Figure 7 displays in addition results of a simple electronic circuit experiment. One observes a monotonic decrease of the control interval. The latter disappears for latencies of the order of 0.1τ. In fact,eq. (11) yields the analytical estimate

$$\frac{\delta}{\tau} < \frac{1-\mathrm{Re}\lambda_+\tau/2}{\mathrm{Re}\lambda_+\tau} \tag{12}$$

for orbits with $\mathrm{Im}\lambda_+\tau = \pi$. Thus acceptable latencies for time–delayed feedback control are related with the Lyapunov exponent of the unstable orbit. Of course such an estimate involves the mean–field–like approximations mentioned previously and gives in general only the correct order of magnitude.

3.3 Explicitly Time–Dependent Control Loops

The introduction of explicit time–dependencies in the control loop changes the control properties considerably as it was already mentioned in section 2 in the context of rhythmic control. Unfortunately there is no systematic theory which can cope with such features, mainly because of the intricate spectral structure of the associated linear stability problems (cf. e.g. [27]). Nevertheless, we are going to explain a simple mechanism by which the system subjected to control optimises its control performance so that control intervals may considerably increase [39]. The following considerations rely on the property that the free system without control is autonomous, so that a Goldstone mode related with time translation invariance occurs.

To explain the essential mechanism we consider the original Pyragas control scheme but suppose that the coupling of the control force to the internal degrees of freedom is mediated by the unstable mode $\boldsymbol{q}_+(t)$

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t)) - K\boldsymbol{q}_+(t)\left[s(t)-s(t-\tau)\right] \quad . \tag{13}$$

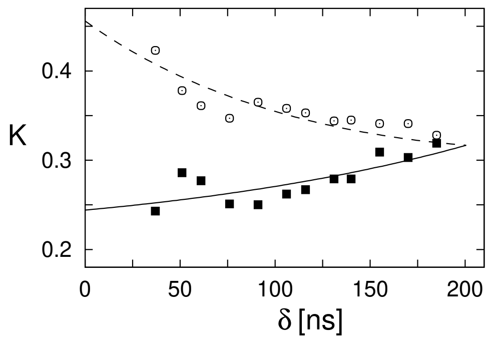

Fig. 7. Control intervals for the original Pyragas scheme in dependence on the control loop latency. Symbols: electronic circuit experiment, lower control threshold (filled squares), upper control threshold (open circles). Lines: analytical result according to eq. (11). See [38] for details of the experiment and the data fit

Such a kind of coupling where eigenmodes are employed is in fact quite common and can be realised e.g. in optical setups [40]. The measured signal $s(t) = g[\boldsymbol{x}(t)]$ depends on the internal state but the actual dependence does not matter. We assume for the simplicity of presentation that the eigenmode is real, but even for the generic case of complex valued modes the arguments follow the same line [41]. Since the system without control force is autonomous the orbit $\boldsymbol{\xi}(t+\delta)$ for arbitrary value of the phase shift δ yields a periodic target state of the system subjected to control, eq. (13). Linear stability analysis of such target states yields the eigenvalue problem

$$\begin{aligned} \Lambda_\alpha \boldsymbol{Q}_\alpha(t) + \dot{\boldsymbol{Q}}_\alpha(t) &= D\boldsymbol{f}(\boldsymbol{\xi}(t+\delta))\boldsymbol{Q}_\alpha(t) \\ &\quad -K\langle dg[\boldsymbol{x}(t)]|\boldsymbol{Q}_\alpha(t)\rangle \left[1-\exp(-\Lambda_\alpha \tau)\right] \boldsymbol{q}_+(t) \quad . \end{aligned} \tag{14}$$

Thus the stability properties of the target states clearly depend on the value of δ. In physical terms the phase δ quantifies the phase difference between the periodic state and the time–dependence introduced by the coupling function $\boldsymbol{q}_+(t)$. It is in general quite difficult to solve eigenvalue problems like eq. (14) by analytical means. But for our very special choice of coupling some analytical results can be obtained. First, for the in–phase orbit i.e. for $\delta = 0$, one realises that the free eigenmode $\boldsymbol{q}_+(t)$ is indeed a solution to eq. (14) provided the inner product $\langle .|.\rangle$ which appears on the right hand side is time–independent. Thus the characteristic equation (5) is recovered for the mode $\nu = +$ and the eigenmode control by eq. (13) is as efficient as diagonal control. Secondly, for $\delta \neq 0$ perturbation expansion may be applied which shows that the control thresholds depends on δ^2 to leading order [39]. Above all, numerical simulations indicate [41] that pronounced dependencies of the control interval on the phase shifts occurs. During an initial transient phase the system may select by self–adaptation an optimal value for the phase shift which finally results in a huge increase of the control performance (cf. Fig. 8). Such a mechanism is not tied to our particular coupling scheme but may occur for general time–dependent control loops in autonomous systems.

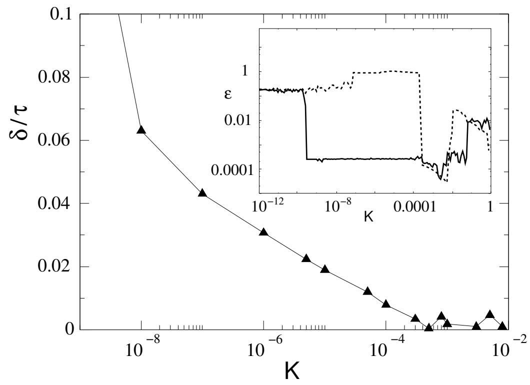

Fig. 8. Dependence of the phase shift on the control amplitude for eigenmode control. Inset: amplitude of the control signal in dependence on the control amplitude for diagonal control (dotted) and eigenmode mode control (full line). Data obtained from a simulation of a reaction–diffusion model with global coupling (cf. [39] for details)

4 Outlook

The previous sections have demonstrated that several aspects of time–delayed feedback control are meanwhile quite well understood from a local point of view, i.e. analytical approaches for the linear stability analysis are available. There remains, however, a considerable gap in understanding how the coupling of the control forces to the internal degrees of freedom and how the properties of the measured signal influence the control performance. Such knowledge would enable us to optimise time–delayed feedback control further. However such an aspect is not the primary goal of time–delayed feedback methods, since these methods are applied when fancy control schemes fail. Whenever e.g. extensive manipulations of the experimental system are possible, fancy data processing can be performed or a mathematical model is available then conventional control techniques borrowed from standard control theory are often superior to time–delayed feedback control. The time–delayed methods just aim at stabilising unknown time–periodic states in systems where only a simple measurement can be performed.

There are of course other aspects of the linear stability which are not fully understood. They are related to the introduction of additional time scales through periodic modulation of the control loop. The surprisingly rich structure of the associated Floquet eigenvalue problem prevents so far a deeper understanding of such control methods. Advances can be expected if a better approach to linear differential–difference equations with time–periodic coefficients becomes available.

Almost nothing is known about global properties of time delay systems, apart from general mathematical statements concerning existence and uniqueness of notions like global attractors, dimensions, Lyapunov exponents, or invariant manifolds [18]. Thus the important question concerning the domain of attraction of particular time–delayed feedback schemes is completely unclear. Such a problem is intimately related to the infinite–dimensional phase space in which the dynamics of differential–difference equations take place. Thus no proper visualisation tools are available. As a first step towards a global analysis concerning domains of attraction one may apply weakly nonlinear bifurcation analysis [18,42] to time–delayed feedback control. For instance, sub– and supercritical behaviour determine domains of attraction of the target state in a characteristic way.

From a more general perspective time delay plays a prominent role in many dynamical systems. Models which received considerable interest in the past refer to biological systems following the lines of the seminal work [43]. Unfortunately no concise picture has developed which constitutes a general dynamical theory of differential–difference equations. By linking the fields of control of chaos with other recent developments of nonlinear dynamics one may consider some nonstandard applications of time–delayed feedback methods. While synchronisation of dynamical systems has already a lot in common with control problems [44,45] recently the effect of time delay has been revived. One quite surprising result concerns the fact that time–delayed feedback can be used for prediction of the dynamical behaviour [46] and such a concept has already been applied in laser experiments [47]. Furthermore the interplay between time delay and noise, albeit of fundamental importance in time–delayed feedback control, is scarcely explored mainly because of the non–Markovian character of the underlying dynamics. Thus the developments are still at its beginning. Even for the simple but famous Kramers problem nontrivial features are observed through the introduction of time scales by time delay [48]. These few remarks already indicate that delay systems are still an undiscovered country which deserves exploration from the theoretical as well as experimental point of view.

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