On–off intermittency associated with the breakdown of one–dimensional motion

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October 25 ,1996

Abstract

The chaotic motion of a particle in a rotationally symmetric Mexican hat potential subjected to a periodic external force is found to undergo a symmetry–breaking instability as the forcing amplitude is increased. Below the threshold the motion is confined to one space dimension, whereas beyond the instability the particle starts to move in both dimensions. At the onset the system exhibits the so–called *on–off intermittency*. The statistical characteristics on both sides of the bifurcation point are studied in comparison with previous investigations of on–off intermittency.

PACS No.: 05.45.+b
Running title: On–Off Intermittency
Submitted to: Phys. Lett. A

1 Introduction

As a highly nonlinear phenomenon intermittency is observed in quite different fields of physics. The most famous example is the hydrodynamical intermittency related to the spatial localization of high vorticity, which causes the irregular motion of the velocity field. Based on the bifurcation theory of equilibrium points Pomeau and

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Manneville classified intermittency into three types (e.g. [1]), which have become popular as "typical routes to chaos".

One decade ago an intermittency mechanism in low dimensional coupled chaotic systems, which is quite different from the Pomeau–Manneville type, was reported [2]. The basic mechanism comes from the competition between the trajectory instability of chaotic elements and the synchronization tendency due to the diffusion–type coupling. The first experimental evidence of this kind of instability, called *type B intermittency* or *intermittency caused by chaotic modulation*, was to our best knowledge observed in an electronic circuit experiment [3].

Recently a closely related type of intermittency was reported in nonlinear dynamical systems different from the above mentioned coupled chaotic oscillator system [4, 5, 6, 7]. It has been termed *on–off intermittency.* Experimental realizations have been performed in coupled electronic circuits too [8, 9] and even in real ferromagnetic resonance experiments [10]. But it turns out that this kind of the intermittency is essentially identical to the previously mentioned type. However, since the terminology on–off intermittency has become popular we also adopt the latter expression here.

It is our objective to propose a new model for on–off intermittency and to report its statistical properties near the transition point.

2 The model system

Consider the motion of a particle in a two dimensional potential with rotational symmetry, subjected to viscous damping and a driving force which acts in a fixed direction

$$\ddot{\boldsymbol{r}}(t) = -\Gamma\dot{\boldsymbol{r}}(t) - \nabla U(\boldsymbol{r}(t)) + F\cos(\Omega t)\boldsymbol{e}_x \quad . \tag{1}$$

Here $\boldsymbol{r}(t) = (x(t), y(t))$ denotes the position of the particle at time t, the potential $U(\boldsymbol{r})$ is assumed to take the Mexican hat form,

$$U(\boldsymbol{r}) = -\frac{a}{2}\|\boldsymbol{r}\|^2 + \frac{b}{4}\|\boldsymbol{r}\|^4, \quad (a, b > 0), \tag{2}$$

Γ denotes the viscous damping constant per unit mass, and F and $\Omega = 2\pi/T$ are respectively the amplitude per unit mass and the frequency of the periodic external force applied in the x-direction with $\boldsymbol{e}_x$ being the corresponding unit vector. In components the equations of motion read

$$\begin{aligned} \ddot{x}(t) &= -\Gamma\dot{x} + [a - b(x^2 + y^2)]x + F\cos(\Omega t) & (3) \\ \ddot{y}(t) &= -\Gamma\dot{y} + [a - b(x^2 + y^2)]y \quad . & (4) \end{aligned}$$

Due to the symmetry of the driven system the equations admit a solution which is entirely bounded to the x–axis, $\boldsymbol{r}(t) = (\xi(t), 0)$ with

$$\ddot{\xi}(t) = -\Gamma\dot{\xi} + [a - b\xi^2]\xi + F\cos(\Omega t) \quad . \tag{5}$$

In particular this type of motion coincides with the one–dimensional dynamics of a particle in a double–well potential. This Duffing oscillator has been studied by many authors from different points of view (e.g. [11, 12, 13]).

Whether this kind of motion is realized in our two–dimensional model depends on its stability with respect to transversal perturbations. For that reason a linear stability analysis of Eqs.(3), (4) according to

$$x(t) = \xi(t) + \delta_{||}(t), \qquad y(t) = \delta_{\perp}(t) \tag{6}$$

is performed. The deviations obey

$$\ddot{\delta}_\mu(t) = -\Gamma\dot{\delta}_\mu(t) + \Xi_\mu(t)\delta_\mu(t), \quad (\mu = ||, \perp) \tag{7}$$

with the abbreviations

$$\Xi_{||}(t) = a - 3b(\xi(t))^2 \tag{8}$$

$$\Xi_{\perp}(t) = a - b(\xi(t))^2 \quad . \tag{9}$$

In polar coordinates

$$\delta_\mu(t) =: L_\mu(t)\cos\theta_\mu(t), \quad \dot{\delta}_\mu(t) =: L_\mu(t)\sin\theta_\mu(t) \tag{10}$$

Eqs.(7) read

$$\dot{L}_\mu(t) = \Lambda^0_\mu(t)L_\mu(t) \tag{11}$$

$$\dot{\theta}_\mu(t) = -\Gamma\sin\theta_\mu(t)\cos\theta_\mu(t) - \sin^2\theta_\mu(t) + \Xi_\mu(t)\cos^2\theta_\mu(t) \quad , \tag{12}$$

where the quantity

$$\Lambda^0_\mu(t) := \sin\theta_\mu(t)[(1+\Xi_\mu(t))\cos\theta_\mu(t) - \Gamma\sin\theta_\mu(t)] \tag{13}$$

governs the exponential growth of the perturbations. Let us define the exponents $\lambda_{||}$ and $\lambda_{\perp}$ as the average values[1]

$$\lambda_\mu := \lim_{t\to\infty}\frac{1}{t}\log\frac{L_\mu(t)}{L_\mu(0)} = \lim_{t\to\infty}\frac{1}{t}\int_0^t \Lambda^0_\mu(s)ds \quad . \tag{14}$$

The exponent $\lambda_{||}$ is identical to the largest Lyapunov exponent of the one–dimensional motion (5), being negative for periodic and positive for chaotic motion. The *transverse Lyapunov exponent* $\lambda_{\perp}$ is relevant for the stability of the one–dimensional solution with respect to perturbations in the y–direction. Namely, the one–dimensional motion is stable for $\lambda_{\perp} < 0$. On the other hand, if $\lambda_{\perp} > 0$, then the one–dimensional solution is unstable and a spatially two dimensional dynamics is eventually observed.

A superficial inspection of Eqs.(7) and (8), (9), especially $\Xi_\perp \geq \Xi_{||}$, might suggest that $\lambda_\perp \geq \lambda_{||}$ holds and hence no stable chaotic one–dimensional motion is possible. But this rough estimate is incorrect in general. Fig. 1 displays the dependence of the exponents on the amplitude of the driving field obtained from a numerical simulation[2]. For $F < F_c(\approx 0.97902)$ one observes a spatially one–dimensional attractor. However, even below this threshold there appears a crisis due to the collision of the one–dimensional chaotic attractor and a coexisting unstable period–seven orbit at $F \approx 0.9788$. At this transition there occurs a sudden increase of the Lyapunov exponent $\lambda_{||}$ which also causes an increase in the negative transverse exponent $\lambda_\perp$. In the neighbourhood of the threshold $F = F_c$ there appears a narrow parameter region where the state sensitively depends on the forcing amplitude. But as the control parameter F is increased beyond this threshold, $\lambda_\perp$ changes its sign from negative to positive values. As a consequence the one–dimensional chaotic motion on the x–axis loses its stability and the particle starts to exhibit a two–dimensional motion in the x–y plane. Fig. 2 displays the perpendicular component of the velocity in dependence on time and the trajectory in real space.

< Fig.1

< Fig.2

3 Intermittency beyond the threshold

At the threshold $F = F_c$ on–off intermittency sets in. We use the Euclidian distance from the former attractor, that means the quantity

$$l(t) = \sqrt{y^2(t) + \dot{y}^2(t)} \tag{15}$$

as a basis for the statistical characterization of the transition. Its rate of change is given by

$$\dot{l}(t) = \Lambda_\perp(t) l(t) \quad , \tag{16}$$

where this expression is merely understood as a definition of the quantity on the right hand side. Of course one may write down an explicit expression for $\Lambda_\perp$ using the full equations of motion, but we refrain from writing down this lengthy expression. When $l(t)$ is sufficiently small, the statistical properties of $\Lambda_\perp(t)$ are well approximated by $\Lambda_\perp^0(t)$ defined in (13). Fig. 3 displays how the coarse–grained transverse expansion rate defined by

$$\bar{\Lambda}_\perp(t) = \frac{1}{T_{cg}} \int_{t-T_{cg}}^{t} \Lambda_\perp(s) ds, \tag{17}$$

T_{cg} being a certain coarse-graining time, evolves in time. One observes bursts, that means abrupt movements in the y–direction, which are related to the sudden change

[1] The averages are understood with respect to a typical initial condition.

[2] We have taken the fixed parameter values $a = 10$, $b = 100$, $\Gamma = 1$, and $\Omega = 3.5$ [12, 13] throughout our simulations. A fourth order Runge–Kutta scheme with step-size $0.01 \times T$ was used for the double precision numerical integration.

of $\bar{\Lambda}_\perp(t)$ from negative to positive values. The presence of the latter fluctuations is thus the necessary condition for the occurrence of on–off intermittency.

As shown in Ref.[14, 15, 16] on–off intermittency near its onset $\lambda_\perp = 0$ turns out to be well described by the so–called multiplicative noise model

$$\Lambda_\perp(t) = \lambda_\perp + f(t) - \beta l(t) \quad . \tag{18}$$

Here the Gaussian–white random force $f(t)$ with $\langle f(t)\rangle = 0$ and $\langle f(t)f(t')\rangle = 2D_\perp\delta(t-t')$ models the fluctuating part of the exponent, whereas the positive constant β takes the dependence on the transversal phase space coordinates into account. Since for small l the expansion rate $\Lambda_\perp$ is well approximated by $\Lambda^0_\perp$, the strength of the fluctuation of $f(t)$ is given by

$$D_\perp = \int_0^\infty \langle(\Lambda^0_\perp(t) - \lambda_\perp)(\Lambda^0_\perp(0) - \lambda_\perp)\rangle dt \tag{19}$$

where the average is a long time or SRB average. Furthermore the stationarity of $l(t)$ demands for $\lambda_\perp > 0$ the condition $\langle\Lambda_\perp(t)\rangle = 0$, which yields $\langle l\rangle = \beta^{-1}\lambda_\perp$. The combination of Eqs.(16) and (18) results in the so–called multiplicative noise model. Its analysis shows that the probability distribution for the intermittency variable $l(t)$ beyond the instability $\lambda_\perp > 0$ takes a power law form for small l [14, 15, 16]

$$P(l) \;\simeq l^{-1+\eta} \quad , \tag{20}$$

where the exponent η is proportional to the transverse Lyapunov exponent

$$\eta = \frac{\lambda_\perp}{D_\perp} \quad . \tag{21}$$

Furthermore the spectral intensity $I(\omega)$, that means the Fourier transform of the autocorrelation function of the intermittency variable, takes a power law form in the low frequency region [14, 15, 16, 7]

$$I(\omega) \simeq \omega^{-\frac{1}{2}} \quad . \tag{22}$$

These statistical features are caused by the self–similar characteristics of the on–off intermittency.

We have evaluated the mentioned quantities numerically for our model system. Fig. 4 displays the probability distribution for $l(t)$. One finds the power law form (20) < Fig.4
and the dependence (21) predicted from the multiplicative noise model. The spectral intensity is shown in Fig.5. One observes the $\omega^{-\frac{1}{2}}$ law in the small frequency region. < Fig.5
The numerical results are thus in good agreement with the theoretical predictions from the multiplicative noise model. Furthermore we have evaluated the distribution

function for the duration times τ of laminar motion, $Q(\tau)$. The result displayed in Fig. 6 clearly shows a power law dependence < Fig.6

$$Q(\tau) \propto \tau^{-3/2} \tag{23}$$

in an intermediate region of τ [5]. From all these features we conclude that the instability in our oscillator displays on–off intermittency beyond the threshold.

4 Subcritical behaviour

Let us now turn to the parameter region where the spatially one–dimensional motion is stable. To investigate the transient behaviour we take initial conditions $(\boldsymbol{r}(0), \dot{\boldsymbol{r}}(0))$ by keeping $l(0) \equiv l_0$ constant, and evaluate the time τ, at which $l(t)$ crosses a certain value $l_c(< l_0)$ for the first time, i.e. $l(\tau) = l_c$. τ will be called the *relaxation time*. In this way we can construct for various initial conditions with constant l_0 the distribution function $W(\tau)$ of relaxation times. Since the underlying dynamics is chaotic, long–lived orbits are supposed to be generated by chaotic saddles. As is well known chaotic saddles yield an exponential decay of the relaxation time distribution for large τ [17]

$$W(\tau) \simeq \exp(-\alpha\tau) \quad , \tag{24}$$

where the decay rate α yields the inverse mean relaxation time. Here it characterizes the statistics of the on–off intermittency below the threshold.

Fig. 7 displays the distributions numerically obtained for a fixed l_0, several values of l_c, and for two values of the forcing amplitude F, (a) appropriately far from F_c, and (b) near F_c. Far from the threshold the exponential decay is independent of l_0 and l_c, while it depends on both quantities near the threshold. < Fig.7

The theoretical description of the preceding section is also capable to relate the decay rate to quantities already determined. Let us for a moment assume that the intermittency variable $l(t)$ is sufficiently small. In this case the transverse expansion rate $\Lambda_\perp(t)$ in (16) can be approximated by (18) by neglecting the l–dependent contribution. Then we obtain the linearized multiplicative noise model,

$$\dot{l}(t) = (\lambda_\perp + f(t))l(t) \quad . \tag{25}$$

This approximation is valid if $|\lambda_\perp| l$ is sufficiently large compared to the nonlinear term. One should remark that after a rescaling of variables, $t = |\lambda_\perp|^{-1}\tilde{t}$ and $f(t) = |\lambda_\perp|\tilde{f}(\tilde{t})$, Eq.(25) depends only on the parameter $D_\perp/|\lambda_\perp|$, which determines the strength of the stochastic force. As shown in Ref.[18] the relaxation time distribution for Eq.(25) is rigorously obtained as

$$W(\tau) = \frac{\log(l_0/l_c)}{\sqrt{4\pi D_\perp}\tau^{3/2}} \exp\left[-\frac{(\lambda_\perp\tau + \log(l_0/l_c))^2}{4D_\perp\tau}\right] \quad . \tag{26}$$

Hence the inverse mean relaxation time reads

$$\alpha = \frac{\lambda_\perp^2}{4D_\perp} \quad , \tag{27}$$

(cf. also Ref.[19]). First of all it should be noted that Eq.(27) shows a characteristic algebraic dependence on the transverse Lyapunov exponent. Furthermore the quantities on the right hand side are solely determined by the underlying attracting set in contrast to ordinary decay rates of repellers. Both features seem to be characteristic for the subcritical regime associated with on–off intermittency. Since we have neglected the nonlinear contributions Eq.(27) is valid except in the immediate vicinity of the transition point. Fig. 7a supports this point of view. Sufficiently far from the transition point the last contribution in Eq.(18) can be neglected and the relaxation process is well described by Eq.(25). But near the transition point $|\lambda_\perp|$ is small and the relaxation process is dominated by both the stochastic force and the nonlinear term. Hence the predictions (26) and (27) become invalid and the decay rate may depend on the cutoff value l_c. This effect is visible in Fig. 7b.

5 Conclusion

Proposing a new model for on–off intermittency we have discussed the intermittency beyond the threshold as well as the transient chaos in the subcritical region. Beyond the transition point the statistical laws such as the distribution of burst amplitudes, the Fourier spectrum, and the laminar length distribution agree quite well with those known so far in other systems. The exponential form of the relaxation time distribution in the deep subcritical regime is compatible with the escape time distribution known from transient chaos [17]. But in the immediate vicinity of the transition point deviations from such a simple behaviour occur. This anomalous behaviour is caused by the interaction of the fluctuation of the local transverse expansion rate $\Lambda_\perp(t)$ with nonlinear contributions to the dynamics in the transverse direction. This new aspect in the transient motion seems to be an ubiquitous phenomenon associated with the onset of on–off intermittency [20].

The model we have investigated in this letter shares the symmetry properties with the experiments of Ref.[10], that means the system without any forcing is rotationally symmetric and has in particular a continuous symmetry group. From this point of view our model seems to be better adapted to such situations compared to the frequently analyzed coupled oscillator models.

Finally we mention that our theoretical approach has been based so far on a stochastic modeling of the chaotic dynamics. However, as shown in Ref.[21, 22] the highly non–Gaussian statistics of intermittent motion is well described by the thermodynamic formalism and the generalized correlation function approach. Work in this direction is in progress and will be reported elsewhere.

Acknowledgments

This study was partially carried out during the stay of H. F. in Marburg as a visiting research fellow supported by the Ministry of Education, Science and Culture, Japan. He thanks Prof. S. Grossmann for his kind advice and various discussions. He is also grateful to Prof. T. Yamada and Drs. H. Hata and S. Miyazaki for evaluating the inverse mean relaxation time near the transition point.

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Figure captions

Fig.1: Lyapunov exponent $\lambda_{||}$ (——) and transverse Lyapunov exponent $\lambda_{\perp}$ (- - - -) of Eqs.(3), (4) in dependence on the forcing amplitude. Lyapunov exponents are given in units of the inverse driving period.

Fig.2: Temporal evolution of the transverse velocity component and trajectories in real space for $F = 0.97925$.

Fig.3: Correlation between the velocity component $\dot{y}(t)$ (——) and the coarse–grained local transverse expansion rate $\bar{\Lambda}_{\perp}(t)$ for $T_{cg} = 1000$ (- - - -) and $F = 0.97925$.

Fig.4: (a) Distribution of the intermittency variable for $F = 0.97908$. The broken line indicates an l^{-1} decay.
(b) Exponent η obtained from least square fits in dependence on the transverse Lyapunov exponent. The broken line indicates the theoretical result (21) with $D_{\perp} = 1.1$ obtained from Eq.(19).

Fig.5: Spectral intensity of the intermittency variable $l(t)$ for $F = 0.97908$. The broken line indicates an $\omega^{-\frac{1}{2}}$ decay, whereas the dotted line shows the ω^{-2} behaviour corresponding to an exponential decay on shorter time scales.

Fig.6: Distribution of laminar periods for $F = 0.97908$. The broken line indicates an $\tau^{-3/2}$ decay. For larger values an exponential decay is visible. Burst and laminar states have been distinguished by the threshold value $l = 10^{-3}$.

Fig.7: Mean relaxation time distributions for (a) $F = 0.9788$, (b) $F = 0.97887$, and several cutoff values l_c, ($l_0 = 1$).