Some basic remarks on eigenmode expansions of time–delay dynamics

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Abstract

We describe in elementary terms how eigenmode expansions can be used to deal with differential–difference equations. As particular applications we present the full analytical solution of linear stochastic time–delay systems and the weakly nonlinear analysis of nonlinear differential–difference equations in the limit of large time delay. Our exposition is essentially based on an explicit analytical expression for the linear spectrum in terms of the Lambert W–function and on the explicit formula for the eigenfunctions of the adjoint equation.

1 Introduction

Time–delay plays an important role in the study of dynamical phenomena. While early and traditional applications of time–delay relate to control problems [1] time–delay dynamics became quite popular among broader communities when it was proposed to play an essential role in modelling biological phenomena [2], such

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as neurodynamics or population dynamics, but also in different fields like laser physics [3]. Time–delay dynamics attracted additional interest in physics when time-delay autosynchronisation was proposed as an efficient method for controlling chaotic systems [4]. Although systems with time–delay have by definition an infinite–dimensional phase space the mathematics is quite straightforward [5], and powerful numerical tools are nowadays available to visualise trajectories and manifolds (cf. e.g. [6]) . However, it seems that even simple analytical techniques, like linear stability analysis, bifurcation theory, and standard perturbation expansions, which are widely used in the context of differential and partial differential equations, are not so much appreciated when it comes to differential–difference equations. Here we try to bridge such a gap. We will recall some basic analytical tools for analysing time–delay dynamics on an elementary level. Most aspects we are presenting here are in principle well known, but we think our presentation makes them available to a wider, less specialised audience.

The first part of our article reviews some basic features of linear differential–difference equations. Albeit these can be dealt with by straightforward Laplace transformation [7], our aim here is to introduce the eigenvalue problem, its analytical solution in terms of the Lambert W–function, and the explicit expression for the adjoint eigenfunction. The latter one turns out to be quite useful in analytical calculations. We skip all mathematical rigour and just focus on formal calculations.

This kind of approach will be used to discuss two different kinds of applications. The first one is stochastic dynamics with time–delay. Such a topic has become quite popular recently, e.g. in the context of analysing tunnelling in multistable stochastic systems [8]. Here we focus on linear stochastic differential–difference equations which, despite the fact that the stochastic dynamics is non–Markovian, can be solved analytically in terms of eigenmodes. Such a type of approach has turned out to be useful in discussing time–delayed feedback control of noise induced oscillations in considerable detail [9]. The second aspect concerns the analysis of nonlinear systems with large time delay in terms of weakly nonlinear perturbation expansions. We demonstrate how standard tools of bifurcation theory and pattern formation are applied to yield amplitude equations describing systems with large delay in a universal way.

2 Eigenmode expansion for linear equations

Let us consider a simple linear differential–difference equation with constant coefficients

$$\dot{x}(t) = -ax(t) + bx(t-\tau) \tag{1}$$

where the initial condition specifies the solution in the whole interval $[-\tau, 0]$, i.e.

$$x(\theta) = \phi(\theta), \qquad -\tau \leq \theta \leq 0 \quad . \tag{2}$$

While most of the subsequent considerations can be generalised to multi–component quantities we will focus now for pedagogical reasons on the scalar case.

In order to determine $x(t)$ one first looks for solutions of exponential type, $x(t) = \exp(\lambda t)$. Then eq.(1) reduces to the nonlinear eigenvalue problem for λ

$$\lambda = -a + b\exp(-\lambda\tau) \quad . \tag{3}$$

This type of equation typically has infinitely many solutions λ_ℓ [7, 5], which in principle can be determined numerically given the values of a and b (cf. [10, 11, 12] for some approaches). It is, however, very convenient to make use of the Lambert W–function, which we introduce in Appendix A, and write down the solutions of (3) explicitly as (cf. [13] and references therein)

$$\lambda_\ell = -a + \frac{W_\ell(b\tau\exp(a\tau))}{\tau} \qquad \ell \in \mathbb{Z} \quad . \tag{4}$$

Following the general wisdom about linear dynamics one suspects that the solution of eq.(1) subjected to the initial condition (2) can be written in terms of a proper linear combination[1]

$$x(t) = \sum_\ell c_\ell \exp(\lambda_\ell t) \quad . \tag{5}$$

Now, the expansion coefficients c_ℓ have to be determined in such a way that eq.(5) satisfies the initial condition (2), i.e.

$$\phi(\theta) = \sum_\ell c_\ell U_\ell(\theta) \tag{6}$$

where we have abbreviated the exponential by the symbol

$$U_\ell(\theta) = \exp(\lambda_\ell \theta) \quad . \tag{7}$$

If we take into account that for $\ell \neq \ell'$ due to eq.(3) the condition

$$\begin{aligned} b\exp(-\lambda_{\ell'}\tau)\int_{-\tau}^{0} \exp(-\lambda_{\ell'}\theta)\exp(\lambda_\ell\theta)\,d\theta &= \frac{b\exp(-\lambda_{\ell'}\tau) - b\exp(-\lambda_\ell\tau)}{\lambda_\ell - \lambda_{\ell'}} \\ &= -1 \end{aligned} \tag{8}$$

is valid, we can easily confirm that the expression[2]

$$V^*_\ell(\theta) = \left[\delta(\theta+0) + b\exp(-\lambda_\ell(\theta+\tau))\right]/N_\ell \tag{9}$$

[1] Concerning the parameters a, b, and τ we exclude choices which yield non generic cases. For instance, the subsequent considerations have to be modified slightly if $b = 0$ or if the eigenvalue equation (3) has degenerated eigenvalues, cf. e.g. [7] for details.

[2] We use the notation $\delta(t+0) = \lim_{\varepsilon\downarrow 0}\delta(t+\varepsilon)$.

is in fact orthogonal to the exponential (7), when the canonical product on $[-\tau, 0]$ is employed. Namely, we have for $\ell' \neq \ell$

$$N_\ell \int_{-\tau}^{0} V_{\ell'}^*(\theta) U_\ell(\theta)\, d\theta = 1 + b \exp(-\lambda_{\ell'}\tau) \int_{-\tau}^{0} \exp(-\lambda_{\ell'}\theta) \exp(\lambda_\ell \theta)\, d\theta = 0 \quad . \tag{10}$$

We choose, whenever possible, the normalisation constant N_ℓ such that a biorthogonal system results,

$$1 = \int_{-\tau}^{0} V_\ell^*(\theta) U_\ell(\theta)\, d\theta \tag{11}$$

that means

$$N_\ell = 1 + b\tau \exp(-\lambda_\ell \tau) = 1 + a\tau + \lambda_\ell \tau \quad . \tag{12}$$

Thus we can easily isolate the coefficient c_ℓ in equation (6) by taking the product with the corresponding "dual eigenfunction" V_ℓ^*

$$\int_{-\tau}^{0} V_\ell^*(\theta) \phi(\theta)\, d\theta = c_\ell \quad . \tag{13}$$

With the help of eqs.(9) and (11) we obtain for the solution (5)

$$\begin{aligned} x(t) &= \sum_\ell \left[\phi(0) + \int_{-\tau}^{0} b \exp(-\lambda_\ell(\theta+\tau)) \phi(\theta)\, d\theta \right] \frac{\exp(\lambda_\ell t)}{N_\ell} \\ &= T(t,0)\phi(0) + \int_{-\tau}^{0} T(t, \theta+\tau) b \phi(\theta)\, d\theta \end{aligned} \tag{14}$$

where we have abbreviated the propagator[3] by

$$T(t,t') = \begin{cases} \sum_\ell \exp(\lambda_\ell(t-t'))/N_\ell & \text{if} \quad t \geq t' \\ 0 & \text{if} \quad t < t' \end{cases} \quad . \tag{15}$$

Eq.(14) clearly shows that the value $\phi(0)$ of the initial condition plays a special role, as it appears isolated as well. Eqs.(14) and (15) imply the normalisation $\lim_{t \to t'} T(t,t') = 1$ which constitutes a kind of nontrivial sum rule for the eigenvalues[4]. Eqs.(14) and (15) are, of course, only meaningful if the system (7) used for expanding the solution (5) is complete. In the special case considered here, i.e., equations with constant coefficients, such a property can be shown in an elementary way using e.g. Laplace transforms [7]. However, for explicitly time–dependent systems the situation may be more subtle (cf. Appendix B).

[3] Depending on the context such a quantity is often called Green's function or fundamental solution as well.

[4] Since the series does not converge uniformly one has to be quite careful when evaluating such limits. In the present case one can show that such a sum rule reduces to $\sum_\ell (1 + a\tau + \lambda_\ell \tau)^{-1} = 1/2$ when no degenerated eigenvalues appear and when the terms of the series are ordered by increasing values of $|\lambda_\ell|$. Apparently this sum rule cannot be obtained from eq.(3) in a simple way. More details can be found e.g. in [7].

3 Inhomogeneous systems

In order to solve a linear inhomogeneous equation

$$\dot{x}(t) = -ax(t) + bx(t-\tau) + f(t) \tag{16}$$

with initial condition (2) one expands the solution in terms of eigenmodes (7) (cf. eq.(5))

$$x(t+\theta) = \sum_{\ell} C_\ell(t) U_\ell(\theta), \quad -\tau \leq \theta \leq 0 \tag{17}$$

where the time–dependence of the expansion coefficients will be chosen to take the inhomogeneous part of the equation of motion into account. Often one refers to such a procedure as the variation of constants formula (cf. [5]). Actually, using the orthogonality (10) and (11) the coefficients are determined by

$$C_\ell(t) = \int_{-\tau}^{0} V_\ell^*(\theta) x(t+\theta)\, d\theta = \left[x(t) + b \int_{t-\tau}^{t} \exp(\lambda_\ell(t-\theta-\tau)) x(\theta)\, d\theta \right] / N_\ell \quad . \tag{18}$$

Taking the derivative with respect to t and using the equation of motion (16) and the characteristic equation (3) we obtain

$$\dot{C}_\ell(t) = \lambda_\ell C_\ell(t) + f(t)/N_\ell \tag{19}$$

where in addition the abbreviation (18) has been employed. With the initial condition $C_\ell(0) = c_\ell$ (cf. eq.(6)) integration yields

$$C_\ell(t) = \exp(\lambda_\ell t) c_\ell + \int_0^t \exp(\lambda_\ell (t-t')) f(t')\, dt'/N_\ell \quad . \tag{20}$$

Thus taking eq.(17) into account we end up with

$$x(t) = T(t,0)\phi(0) + \int_{-\tau}^{0} T(t,\theta+\tau) b\phi(\theta)\, d\theta + \int_0^t T(t,t') f(t')\, dt' \tag{21}$$

where eqs.(9), (13), and (15) have been employed.

4 Linear stochastic delay dynamics

Linear stochastic delay systems have been, of course, analysed in detail in the literature (cf. e.g. [14] for a proper mathematical treatment or [15] for an analytical treatment of linear stochastic delay systems with a Fokker–Planck approach). Here, we just treat such systems as an application of the previous considerations. Suppose an additive stochastic force enters the delay dynamics, i.e. consider eq.(16) with the choice

$$f(t) = \sqrt{D}\xi(t) \quad . \tag{22}$$

Here $\xi(t)$ denotes Gaussian white noise with normalised correlation function $\langle\xi(t)\xi(t')\rangle = 2\delta(t-t')$. The diffusion constant D determines the strength of the stochastic force. We treat the stochastic force like an ordinary function, i.e. we will consider the limit of δ–correlations in the final result of our computation. Thus we adopt a Stratonovich point of view, although we are not introducing stochastic integrals in a formal way. For the initial condition we make for simplicity the deterministic choice eq.(2).

Evaluating eq.(17) at $\theta = 0$ gives

$$x(t) = \sum_\ell C_\ell(t). \tag{23}$$

From eqs.(19) and (22) we see that each of the modes C_ℓ obeys a complex valued Ornstein–Uhlenbeck equation

$$\dot{C}_\ell(t) = \lambda_\ell C_\ell(t) + \frac{\sqrt{D}}{N_\ell}\xi(t) \quad , \tag{24}$$

with initial condition $C_\ell(0) = c_\ell$. Thus the stochastic process $x(t)$ can be formally written as an infinite sum of Ornstein–Uhlenbeck processes $C_\ell(t)$, which are driven by a common noise source $\xi(t)$. However, such an approach is of limited use as there does not exist a simple closed equation of motion for the statistical properties of the variable $x(t)$ alone. Nevertheless, eq.(23) tells us that $x(t)$ is still a Gaussian variable. Thus its properties are entirely determined by the first two cumulants, $\langle x(t)\rangle$ and $\langle x(t)x(s)\rangle$. For the expectation value we obtain as expected the deterministic part of the motion (cf. eqs.(21) and (22))

$$\langle x(t)\rangle = T(t,0)\phi(0) + \int_{-\tau}^{0} T(t,\theta+\tau)b\phi(\theta)\,d\theta \tag{25}$$

while for the second cumulant we have

$$\begin{aligned}\langle\delta x(t)\delta x(s)\rangle &= D\int_0^t\int_0^s T(t,t')T(s,s')\langle\xi(t')\xi(s')\rangle\,ds'dt' \\ &= 2D\int_0^s T(t,t')T(s,t')\,dt' \quad \text{if } t\geq s \quad .\end{aligned} \tag{26}$$

Here we have abbreviated the fluctuations by $\delta x(t) = x(t) - \langle x(t)\rangle$.

If we consider a stable system, i.e. a propagator (15) where all the eigenvalues have negative real part, $\mathrm{Re}\lambda_\ell < 0$, then by taking the long time limit in eq.(25) and (26) we obtain the stationary expectation value and two–time autocorrelation function. Obviously $\langle x\rangle_{st.} = 0$. For the autocorrelation function we have for $t \geq 0$, using eq.(15)

$$\langle\delta x(t)\delta x(0)\rangle_{st} = \lim_{s\to\infty}\langle\delta x(t+s)\delta x(s)\rangle$$

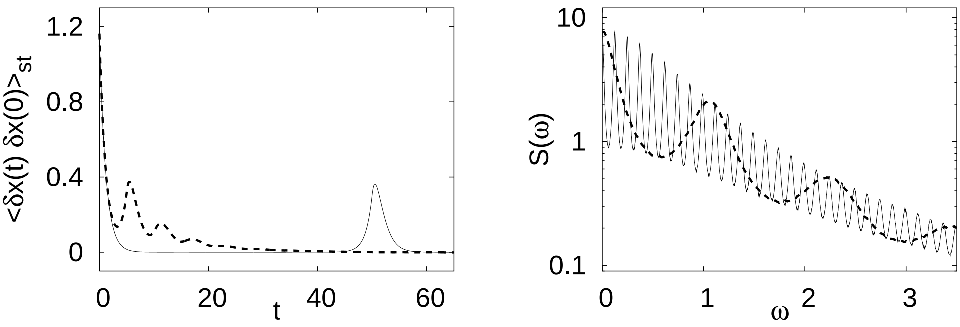

Figure 1: Numerical simulation of the linear stochastic delay system (16), (22) for $a = 1$, $b = 1/2$, $D = 1$ and two different values of the delay. Left: Correlation function for $\tau = 5$ (thick broken line) and $\tau = 50$ (solid line). Right: Corresponding power spectra for $\tau = 5$ (thick broken line) and $\tau = 50$ (solid line) in the low frequency domain.

$$
\begin{aligned}
&= 2D \int_0^\infty T(t, -t'')T(0, -t'')\, dt'' \\
&= 2D \sum_{\ell\ell'} \frac{\exp(\lambda_\ell t)}{N_\ell N_{\ell'}(-\lambda_\ell - \lambda_{\ell'})} \quad . \qquad (27)
\end{aligned}
$$

The correlation function clearly consists of a superposition of decaying exponentials. Accordingly, the power spectrum, i.e. the Fourier transform, is a sum of Lorentzians where the resonance frequencies and the linewidths are determined by the eigenvalues λ_ℓ (cf. figure 1). Actually, the power spectrum can be derived in closed analytical form by Fourier transformation (cf. [16]).

The correlation function clearly recovers after the delay time has elapsed. But especially for large delay τ a pronounced exponential decay appears on the scale $t \lesssim \tau$. Such a decay is clearly not related with a single eigenvalue. It can be understood qualitatively if we have a closer look at the whole spectrum λ_ℓ. For large delay and $a > 0$ we may adopt eq.(50). Using a Taylor series expansion of eq.(50) with respect to $2\pi i\ell/\tau$ we have

$$\lambda_\ell = \lambda_0 + v\frac{2\pi i\ell}{\tau} + \Delta\left(\frac{2\pi i\ell}{\tau}\right)^2 + \dots \qquad (28)$$

where the expansion coefficients are given by

$$\lambda_0 = \frac{1}{\tau}\ln\frac{b}{a + \ln(b\tau)/\tau}, \quad v = 1 - \frac{1}{a\tau + \ln(b\tau)}, \quad \Delta = \frac{1}{2\tau(a + \ln(b\tau)/\tau)^2} \quad . \qquad (29)$$

The spectrum (28) may be written in terms of a "wavenumber" $q = 2\pi\ell/\tau$ and thus resembles spectra that occur in dissipative spatially extended systems. Employing eq.(12) we obtain from eq.(27) in the limit of large delay

$$\langle \delta x(t)\delta x(0)\rangle_{st} \simeq \frac{D}{2\pi^2}\int\int \frac{\exp(\lambda(q)t)}{(a+\lambda(q))(a+\lambda(q'))(-\lambda(q)-\lambda(q'))}\,dq\,dq' \qquad (30)$$

where $\lambda(q)$ abbreviates the "dispersion relation" determined by eq.(28). For a qualitative estimate we adopt the approximation $\lambda(q) = \lambda_0 + iq$ with λ_0 being negative of order $\mathcal{O}(1/\tau)$. The integrals may be estimated in leading order of $1/\tau$ in a straightforward manner. Usual contour integration yields in lowest order of $1/\tau$

$$\int \frac{dq'}{(a+\lambda(q'))(-\lambda(q)-\lambda(q'))} \sim \frac{2\pi}{a-iq} \quad . \qquad (31)$$

Hence, eq.(30) results in

$$\langle \delta x(t)\delta x(0)\rangle_{st} \sim \frac{D}{\pi}\int \frac{\exp(iqt)}{(a+iq)(a-iq)}\,dq = \frac{D}{a}\exp(-a|t|) \quad . \qquad (32)$$

Thus, the quasicontinuous character of the spectrum (28) in the limit of large delay τ causes the initial decay of correlations, a property quite well known in e.g. statistical physics. However, our qualitative estimate for $\lambda(q)$ is too crude to yield the correct asymptotic behaviour. The latter can be obtained rather straightforwardly either using the analytical expression of the power spectrum or the analytical expression for the correlation on $t \in [0,\tau]$ (cf. e.g. [14]). The correct asymptotic behaviour reads $D\exp(-\sqrt{a^2-b^2}t)/\sqrt{a^2-b^2}$. However, our qualitative argument shows that the quasicontinuous character of the spectrum caused by large time delay is crucial to understand the dynamics in the limit of large delay times.

5 Weakly nonlinear analysis

So far we have focussed solely on linear systems. Nonlinear systems with slow modes, like dynamics close to an instability, can be dealt with in terms of a weakly nonlinear perturbation expansion [5]. We illustrate this concept by considering the model system (cf. also [17])

$$\dot{x}(t) = -ax(t) + bx(t-\tau) - gx^3(t) \quad . \qquad (33)$$

Suppose that some of the eigenvalues λ_k of the linear part have small real part in modulus for say $|k| < k_c$, and that all the other eigenvalues have negative real part. Then one can expand the solution of the nonlinear equation in terms of these so called critical modes

$$x(t+\theta) = \sum_{|k|<k_c} C_k(t)U_k(\theta) + R_2(t,\theta) \qquad (34)$$

where the summation is restricted to the critical range $|k| < k_c$ and the remainder R_2 is supposed to be of second order in the coefficients C_k. In contrast to eq.(17) only a subset of modes enters the expansion. Actually eq.(34) describes a surface within the full phase space of the differential–difference equation which is tangential to the plane spanned by the critical modes. There is some freedom in the choice of the coordinates $C_k(t)$ of the surface. Such ambiguity can be used to require that the remainder R_2, describing the curvature of the surface, is orthogonal to the critical plane. Thus one may choose

$$0 = \int_{-\tau}^{0} V_k^*(\theta) R_2(t, \theta)\, d\theta \tag{35}$$

where it is from now on assumed that the index k is restricted to the critical range $|k| < k_c$. Eqs.(34) and (35) imply that the expansion coefficients $C_k(t)$ obey the relation (18). Taking the time derivative and using the nonlinear evolution equation (33) we obtain (cf. eq.(19))

$$\dot{C}_k(t) = \lambda_k C_k(t) - g x^3(t)/N_k \quad . \tag{36}$$

If we use eq.(34) at $\theta = 0$ to express the nonlinear contribution in eq.(36), take into account that the remainder R_2 is of second order, and restrict the resulting amplitude equation to contributions of third order we end up with[5]

$$\dot{C}_k(t) = \lambda_k C_k(t) - \gamma_k \sum_{k_1,k_2,k_3} C_{k_1}(t) C_{k_2}(t) C_{k_3}(t) \tag{37}$$

where the coupling coefficient is given by

$$\gamma_k = g/N_k = \frac{g}{1 + b\tau \exp(-\lambda_k \tau)} \quad . \tag{38}$$

At this stage the time delay has disappeared and the dynamics has been reduced to an equation for the amplitudes C_k. The accuracy of such an expansion may be demonstrated with a simple numerical example. If we choose for instance $a = 0.1$, $b = -0.5$, $g = 0.5$, and $\tau = 4$ then the eigenvalue equation (3) which corresponds to eq.(33) yields a single complex conjugate pair of eigenvalues $\lambda_0 = \lambda_{-1}^* = 0.01575 + i0.45498$ while all the other eigenvalues have negative real part. Thus just two modes $k \in \{0, 1\}$ are relevant for the expansion (34). Eq.(37) yields a two–dimensional ordinary differential equation. Figure 2 shows that the solution of eq.(33) and the result obtained from eq.(37) differ by less than 5%. More details about the application of such centre manifold reductions to this particular example can be found e.g. in [17].

[5] Since in our example only cubic nonlinearities appear we do not need to calculate the explicit expression for $R_2(t, \theta)$. In the case of additional quadratic nonlinear terms one may compute R_2 from the invariance condition $dx(t+\theta)/dt = dx(t+\theta)/d\theta$ using an expansion up to quadratic terms in the amplitudes, $R_2(t, \theta) = \sum_{kk'} \rho_{kk'}(t, \theta) C_k(t) C_{k'}(t)$.

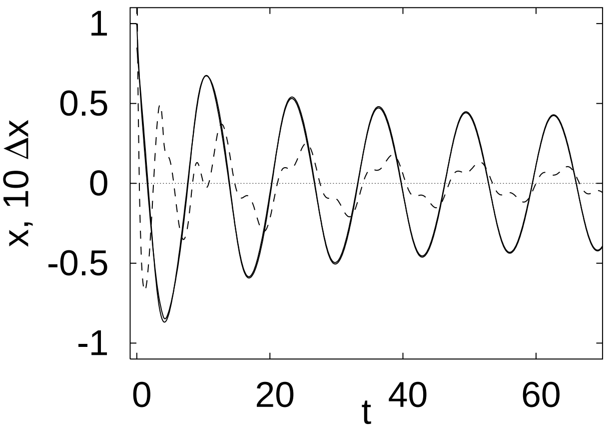

Figure 2: Numerical simulation of the nonlinear differential difference equation (33) for $a = 0.1$, $b = -0.5$, $g = 0.5$, and $\tau = 4$ with initial condition $\phi(\theta) = 1$ and of the corresponding amplitude equation (37) with initial condition $C_{k=0}(0) = C^*_{k=-1}$ according to eqs.(13) and (9) (solid lines). The two solid lines can hardly be distinguished on the scale used. The difference of both solutions, multiplied by a factor of ten, is displayed as well (broken line).

Some general properties of such an approach may be discussed when we focus on the limit of large time delay. In the limit of large delay τ the number of modes becoming critical scales linearly with τ when $b > a > 0$ (cf. eqs.(28) and (29)). Actually, such a kind of high–dimensional dynamics is characteristic for delay systems with large delay times (cf. e.g. [18]). As already mentioned in the previous section the spectrum of the linear part resembles the structure known from spatially extended dissipative systems. Thus we may follow standard procedures developed in the context of pattern formation [19].

Since the real part of the eigenvalues is small (cf. eq.(50)) following the usual spirit of normal form analysis non–resonant contributions can be eliminated from the amplitude equation (37) by applying a suitable nonlinear transformation. Introducing new amplitudes through a Taylor series expansion

$$\Gamma_k(t) = C_k(t) + \sum_{k_1,k_2,k_3} \alpha_{k,k_1,k_2,k_3} C_{k_1}(t) C_{k_2}(t) C_{k_3}(t) + \ldots \tag{39}$$

eq.(37) yields up to the third order

$$\begin{aligned} \dot{\Gamma}_k(t) &= \lambda_k \Gamma_k(t) \\ &- \sum_{k_1,k_2,k_3} [\gamma_k - (\lambda_{k_1} + \lambda_{k_2} + \lambda_{k_3} - \lambda_k)\alpha_{k,k_1,k_2,k_3}] \Gamma_{k_1}(t)\Gamma_{k_2}(t)\Gamma_{k_3}(t) \quad . \end{aligned} \tag{40}$$

As long as $\lambda_{k_1} + \lambda_{k_2} + \lambda_{k_3} - \lambda_k$ does not vanish the corresponding nonlinear term in eq.(40) can be eliminated by an appropriate choice of the transformation parameter, $\alpha_{k,k_1,k_2,k_3} = \gamma_k/(\lambda_{k_1}+\lambda_{k_2}+\lambda_{k_3}-\lambda_k)$. By such an approach the equation of motion can be cast into its simplest (so–called normal) form. To evaluate the resonance condition $\lambda_k = \lambda_{k_1} + \lambda_{k_2} + \lambda_{k_3}$ we apply the simplest approximation, i.e. $\lambda_k = 2\pi ik/\tau$. Then all nonlinear contributions obeying $k \neq k_1 + k_2 + k_3$ can be eliminated from eq.(40) and we obtain

$$\dot{\Gamma}_k(t) = \lambda_k \Gamma_k(t) - \gamma_k \sum_{k_1+k_2+k_3=k} \Gamma_{k_1}(t)\Gamma_{k_2}(t)\Gamma_{k_3}(t) \quad . \tag{41}$$

For the coupling coefficient γ_k eq.(38) yields with the same type of approximation the result

$$\gamma_k = \frac{g}{1 + a\tau + \lambda_k \tau} \simeq \frac{g/\tau}{a + 2\pi ik/\tau} \quad . \tag{42}$$

Evaluation of the leading linear term in eq.(41) requires greater care for the spectrum. We therefore apply eq.(28) that has been obtained by a Taylor series expansion of of eq.(50) with respect to $2\pi ik/\tau$. The spectrum (28) clearly shows, with respect to the "wavenumber" $2\pi ik/\tau$, the parabolic shape which is well known from instabilities in spatially extended systems. This type of analogy may be pursued further. If one introduces a (real valued) spatio–temporal field through the definition

$$\Phi(t,\theta) = \sum_k \Gamma_k(t) \exp(2\pi ik\theta/\tau) \tag{43}$$

where θ denotes the "spatial" variable on $\theta \in [-\tau, 0]$ then, according to eq.(41) the field obeys the partial differential equation

$$\frac{\partial \Phi}{\partial t} = \lambda_0 \Phi(t,\theta) + v\frac{\partial \Phi}{\partial \theta} + \Delta \frac{\partial^2 \Phi}{\partial \theta^2} - \frac{g}{\tau}\int_{-\tau}^{0} \exp(a\theta')\Phi^3(t, \theta + \theta')\, d\theta' \tag{44}$$

if we take the expressions (42) and (28) into account. In writing down eq.(44) we assume periodic boundary conditions for θ, according to the definition (43). The condition $a\tau \gg 1$ as been employed as well to obtain a simple expression for the kernel.

The description of the time–delay system in terms of eq.(44) is in fact quite intuitive. First of all the field introduced by eq.(43) is directly linked to the time–dependent solution of the delay system. Comparing the expansions (34) and (43) one recognises that $x(t + \theta)$ is approximately given by $\Phi(t, \theta)$ when the leading order asymptotics in τ of the eigenvalues is used, $\lambda_k = 2\pi ik/\tau$, and the nonlinear corrections due to the coordinate transformation (39) are ignored. Actually such a kind of spatio–temporal representation of delay dynamics, where the history is considered to be a spatial variable $\theta \in [-\tau, 0]$, has been already proposed quite a while ago to analyse the dynamics of delay systems [20]. Secondly, the structure of eq.(44) may be understood quite easily as well. The linear contributions just take

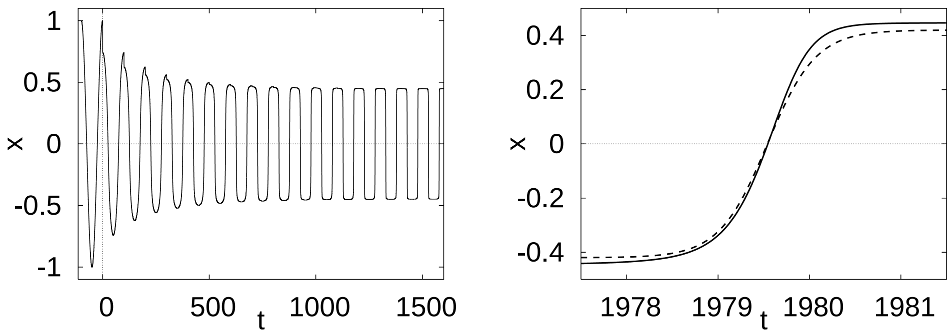

Figure 3: Left: Numerical simulation of the nonlinear differential difference equation (33) for $a = 4$, $b = 5$, $g = 5$, and $\tau = 100$ with initial condition $\phi(\theta) = \cos(2\pi\theta/\tau)$. Right: Detailed view of the numerical solution (solid line) and of the kink solution of the amplitude equation (cf. eq.(44)) (broken line) in the stationary state.

the properties of the spectrum into account. In particular, the convective term represents the temporal shift of the history, which occurs when integrating the original time–delay dynamics. Furthermore, the nonlocal nonlinear contribution is just the consequence of the integration of the nonlinear part, where the kernel is caused by the instantaneous linear part of the equation of motion. Overall, eq.(44) may be simplified further. The convective contribution can be eliminated by transformation to the co–moving frame. Then all remaining coefficients are of order $\mathcal{O}(1/\tau)$ and the equation of motion captures the essential long time dynamics, in particular on time scales $t \gg \tau$. But such a kind of analysis is beyond the scope of the present contribution.

Nevertheless, one can get an impression about the accuracy of our approximation scheme by just looking at numerical simulations of eq.(33) (cf. figure 3). If one takes an oscillating initial condition then the solution develops a kink–antikink like characteristic when $b > a > 0$ and when a large time delay is employed. Such a type of behaviour is in fact quite typical for real Ginzburg Landau equations like eq.(43) (cf. e.g. [21] and references therein). Actually, if one approximates the integral contribution in eq.(43) by a local expression, i.e. by $g/(a\tau)\Phi^3(t,\theta)$, the kink can be expressed easily in analytical terms by $\sqrt{\lambda_0 a\tau/g}\tanh(\sqrt{\lambda_0/(2\Delta)}\theta)$. Such an expression fits quite well with the direct simulation of the time–delay dynamics (cf. figure 3). Thus the reduction to the normal form yields reasonable results even in the limit of large delay time. One should, however, mention that the accuracy of the approximation decreases if $b \gg a$. But from a qualitative perspective the approximation scheme still produces reasonable results.

The derivation of amplitude equations, which are well established in the context of pattern formation in large aspect ratio systems, is in fact a standard tool for time–delay dynamics as well. For instance, differential–difference equations display in the vicinity of Hopf bifurcations features known from hard–mode instabilities when the limit of large time–delay is considered. The corresponding effective equation of motion, i.e. a complex Ginzburg–Landau equation, has been derived by multiple scaling techniques [22]. The latter are known to be equivalent to centre manifold reductions presented here [23]. While these approaches are based on a perturbation expansion close to a bifurcation point the present calculation shows that one might be able to relax such a condition to some extent. Our resulting amplitude equation is essentially based on an expansion for large τ only. Thus one may expect that normal form calculations in differential–difference equations have a wider range of applicability when the limit of large time delay is considered.

6 Conclusion

Deterministic nonlinear delay differential equations arise in particular in the context of time–delayed feedback control of chaotic systems, as originally proposed by Pyragas [4] in the context of chaos control, or more generally, as a powerful scheme for stabilizing unstable periodic orbits or space-time patterns (cf. e.g. [24, 25]).

The full analytical solutions in terms of an eigenmode expansion which we have presented in this paper allows for interesting applications to time–delayed feedback control. In a linearized approximation of the dynamics of noise–induced oscillations [9, 26], for instance, one can obtain explicit expressions for the autocorrelation function and the correlation time. Hence predictions how control affects the coherence properties of the noise–induced dynamics become available.

Weakly nonlinear perturbation expansions are able to provide insight into global properties of time–delayed feedback control. Within a recent study [27, 28] these concepts have been successfully applied to determine universal features of basins of attractions of periodic orbits subjected to time–delayed feedback control.

A combination of both aspects is supposed to yield new analytical insight into time–delayed feedback control of chaos in particular, and into the dynamics of nonlinear stochastic time–delay dynamics in general.

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A The Lambert W–function

The Lambert W–function is the multivalued inverse of $f(z) = z\exp(z)$, i.e.

$$z = W_\ell(z)\exp(W_\ell(z)), \qquad \ell \in \mathbb{Z} \quad . \tag{45}$$

The main branch $W_0(z)$ has a branch point at $z = -1/e$ and a branch cut along the negative real axis $-\infty < z < -1/e$. It is real valued along the upper edge of the branch cut and otherwise real valued along the real axis. $W_1(z)$ and $W_{-1}(z)$ have branch points at $z = 0$ and $z = -1/e$ with branch cuts along the negative real axis, $-\infty < z < 0$ and $-\infty < z < -1/e$. $W_{-1}(z)$ is real valued along the upper edge of the cut $-1/e < z < 0$. Along its branch cut $W_0(z)$ connects to $W_1(z)$ and $W_{-1}(z)$, respectively. The remaining branches $W_\ell(z)$, $|\ell| \geq 2$ have a single branch point at $z = 0$ and a branch cut along the negative real axis connecting subsequent branches. A comprehensive summary of the properties of the Lambert W–function can be found in [13].

In particular, the main branch of the Lambert W–function possesses the series expansion

$$W_0(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n \tag{46}$$

which converges in the disk $|z| < 1/e$. The non–principal branches admit the asymptotic expansion

$$W_\ell(z) \simeq \ln(z) + 2\pi i\ell - \ln(\ln(z) + 2\pi i\ell), \qquad (\ell \neq 0) \tag{47}$$

for $z \to 0$ as well as $|z| \to \infty$. In eq.(47) the logarithm is defined through its main branch with a branch cut along the negative real axis, i.e. $\ln(r\exp(i\varphi)) = \ln r + i\varphi$, $-\pi < \varphi \leq \pi$. Eq.(47) is valid for the principal branch, $\ell = 0$, as well in the asymptotic limit $|z| \to \infty$.

If we rewrite the characteristic equation (3) as

$$(\lambda + a)\tau = b\tau\exp(a\tau)\exp(-(\lambda + a)\tau) \tag{48}$$

then it becomes obvious that the solutions can be expressed in terms of the Lambert W–function as

$$\lambda_\ell = -a + \frac{W_\ell(b\tau\exp(a\tau))}{\tau} \qquad \ell \in \mathbb{Z} \quad . \tag{49}$$

For large values of the argument the asymptotic property (47) yields a simple analytic formula for the eigenvalue spectrum

$$\lambda_\ell\tau \simeq \ln(b\tau) + 2\pi i\ell - \ln(a\tau + \ln(b\tau) + 2\pi i\ell), \qquad |b\tau\exp(a\tau)| \gg 1 \tag{50}$$

which, e.g., describes the spectrum of systems with large delay τ for $a > 0$.

B Small solutions

Consider the explicitly time–dependent linear differential–difference equation

$$\dot{x}(t) = \cos(t)x(t-2\pi) \tag{51}$$

with initial condition

$$x(0) = \phi(0) = 1, \qquad x(\theta) = \phi(\theta) = 0 \quad -2\pi \leq \theta < 0 \quad . \tag{52}$$

When looking for eigenmodes of eq.(51) we have to use the Floquet ansatz $\exp(\lambda t)q_\lambda(t)$ where $q_\lambda(t) = q_\lambda(t+2\pi)$ takes the periodic time–dependence of the coefficient in eq.(51) into account. Then the equation of motion yields

$$\dot{q}_\lambda(t) + \lambda q_\lambda(t) = \cos(t)q_\lambda(t) \quad . \tag{53}$$

There is actually only a single solution to this eigenvalue problem, namely

$$\lambda = \frac{1}{2\pi}\int_0^{2\pi} \cos(t)\,dt = 0, \qquad q_{\lambda=0}(t) = \exp(\sin(t)) \quad . \tag{54}$$

The only solution which can be constructed from the eigenmode and which fulfils the initial condition at $\theta = 0$ thus reads

$$x_\infty(t) = \exp(\sin(t)) \quad . \tag{55}$$

But there is no guarantee that this expression really complies with the whole initial condition (52).

Eqs.(51) and (52) can be solved by iteration. Since by the initial condition $x(t-2\pi) = 0$ holds for $0 \leq t < 2\pi$, eq.(51) yields $x(t) = 1$ if $0 \leq t < 2\pi$. Proceeding in such a way one obtains

$$x(t) = \sum_{k=0}^{n} \frac{\sin^k(t)}{k!} \qquad \text{if } 2\pi n \leq t < 2\pi(n+1) \quad . \tag{56}$$

In fact, the solution (56) differs from the "approximate" expression (55) by the amount

$$R(t) = x_\infty(t) - x(t) = \sum_{k=n+1}^{\infty} \frac{\sin^k(t)}{k!} \qquad \text{if } 2\pi n \leq t < 2\pi(n+1) \quad . \tag{57}$$

This remainder decays faster than exponentially when t increases. One has to be aware of such so–called "small solutions" when one solves differential–difference equations with expansion in terms of eigenmodes. In general, the eigensystem is not complete. Fortunately, such a shortcoming does not matter as long as the long time dynamics is at stake.

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