Dynamics of the stochastic Duffing oscillator in Gaussian approximation *

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Abstract

The previously developed general concept of the Gaussian approximation allows the description of the driven stochastic Duffing oscillator by ordinary differential equations. These equations are analysed analytically as well as numerically to investigate the modification of simple bifurcations and of the Feigenbaum cascade by stochastic forces. Furthermore the method allows a qualitative discussion of the stochastic system in the deterministic irregular region.

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1 Introduction

Due to the development of nonlinear dynamics in the last two decades the influence of random forces on nonlinear dynamical systems has grown in interest [1, 2]. As the deterministic motion is only partially understood the results on stochastic systems are rare even in the limit of small Gaussian random forces. Most investigations in that field were made towards the computation of stationary solutions for autonomous systems [3]. To get some insight into the time dependent dynamics we have developed a scheme [4] to describe the dynamics approximately by a low dimensional set of ordinary differential equations, which is easier to handle as the stochastic description analytically as well as numerically.

To clarify the basic idea let us consider as a model system the one dimensional motion of a particle in a potential $U(x,t)$ subjected to damping and a Gaussian white noise. The dynamics is given by the Fokker– Planck equation

$$\frac{\partial \rho}{\partial t} = -\mathbf{\Lambda}(t)\rho = \left(-\frac{\partial}{\partial x}v + \frac{\partial}{\partial v}(U'(x,t)+\gamma v) + \frac{\partial^2}{\partial v^2}D\right)\rho(x,v,t) \quad . \qquad (1)$$

Assuming sufficiently small diffusion one may hope that the motion can be described by a set of ordinary differential equations as in the deterministic case. We have shown that this goal can be achieved by the following set for the relevant expectation values $\langle x\rangle$, $\langle v\rangle$, Δx^2, Δvx, Δv^2 [1]

$$\begin{aligned}
\dot{\langle x\rangle} &= \langle v\rangle \\
\dot{\langle v\rangle} &= -U'(\langle x\rangle,t) - \sum_{\nu=1}^{\infty}\frac{1}{2^\nu \nu!}U^{(2\nu+1)}(\langle x\rangle,t)(\Delta x^2)^\nu - \gamma\langle v\rangle \\
\dot{\Delta x^2} &= 2\Delta vx \\
\dot{\Delta vx} &= \Delta v^2 - \Delta x^2\sum_{\nu=0}^{\infty}\frac{1}{2^\nu \nu!}U^{(2\nu+2)}(\langle x\rangle,t)(\Delta x^2)^\nu - \gamma\Delta vx \\
\dot{\Delta v^2} &= -2\Delta vx\sum_{\nu=0}^{\infty}\frac{1}{2^\nu \nu!}U^{(2\nu+2)}(\langle x\rangle,t)(\Delta x^2)^\nu - 2\gamma\Delta v^2 + 2D \quad . \qquad (2)
\end{aligned}$$

[1] $\delta x := x - \langle x\rangle$, $\Delta x^2 = \langle(\delta x)^2\rangle$, etc.

Roughly spoken these equations follow from eq.(1) by approximating the (localized) distribution function self consistently by a Gaussian.

$$\rho(x,v,t) \to \mathcal{R}(x,v,t) = \frac{1}{Z}\exp\left(-\frac{\Delta v^2(\delta x)^2 - 2\Delta vx(\delta v)(\delta x) + \Delta x^2(\delta v)^2}{2(\Delta x^2\Delta v^2 - (\Delta vx)^2)}\right) \tag{3}$$

For further details on the derivation, structure and properties of eqs.(2) the reader should consult [4].

In the next section I will investigate analytically the well known weakly driven Duffing oscillator. For this simple system one is able not only to treat eqs.(2) but also the Fokker– Planck equation (1) perturbativly, so that the validity of eqs.(2) can be checked by comparison. Special attention is focussed onto the Codimension 2 bifurcation point of the deterministic system. As the main point of this article we discuss numerically in the third section the dynamics of the strongly driven stochastic Duffing oscillator with the help of eqs.(2). For this system an approach by the full Fokker– Planck equation seems to be impossible. Especially the modification of the period doubling scenario is investigated in detail. To my knowledge it is the first time that the conclusions are drawn for the full stochastic system and not for a simple stochastic mapping [5]. Finally the results will be summarized in a conclusion.

2 The weakly driven Duffing oscillator

I consider the weakly driven dynamics of a particle in an anharmonic potential subjected to small damping. Furthermore the discussion will be restricted to the case of first order resonance. In terms of formulas these circumstances are expressed as

$$\begin{aligned} U(x,t) &= \frac{\omega_0}{2}x^2 + \epsilon(\frac{1}{3}x^4 + xh\cos t) \\ \epsilon\bar{\gamma} := \gamma = O(\epsilon) & \qquad \epsilon\delta := 1-\omega_0^2 = O(\epsilon) \quad , \end{aligned} \tag{4}$$

where h denotes the amplitude of the driving field. It is assumed that the frequencies of the external perturbation and of the harmonic part of the potential deviate by a small amount which is of the same order of magnitude as the damping constant γ and scales with the expansion parameter ϵ. To ensure the stability of the stochastic system we require that the diffusion constant scales with ϵ too

$$\epsilon \bar{D} := D = O(\epsilon) \quad . \tag{5}$$

The discussion of the deterministic dynamics ($\bar{D} = 0$) can be achieved by application of the standard averaging method or aquivalent approaches. As the detailed analysis can be found in different textbooks [6, 7] I only present the results as far as needed for further computations. With the standard transformation

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ -\sin t & -\cos t \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \tag{6}$$

to variables ξ, η varying only appreciably on the slow time scale $\tau := \frac{\epsilon}{2} t$ one can change the original deterministic equations (cf. eq.($2_{1,2}$)) to an autonomous system. If the bifurcation parameters (δ, h) take values $\delta_c = \sqrt{3}\,\bar{\gamma}$, $h_c = (\frac{2}{3}\bar{\gamma})^{\frac{3}{2}}$ this system possesses a Codimension 2 bifurcation point (cusp point) that represents the onset of bistability. The corresponding marginally stable fixed point is given by $\xi_0 = \sqrt{\frac{\delta_c}{6}}, \eta_0 = \sqrt{\frac{\delta_c}{2}}$. The coordinates adapted to the stable and center manifold are determined by

$$\begin{aligned} \xi &= \sqrt{\delta_c}\left(\frac{1}{\sqrt{6}} + \frac{z}{\sqrt{3}} + y\right) \\ \eta &= \sqrt{\delta_c}\left(\frac{1}{\sqrt{2}} - z\right) \end{aligned} \tag{7}$$

and the equation of the center manifold reads

$$y = -\frac{4}{\sqrt{6}} z^2 + O(z^3) \quad . \tag{8}$$

→ Fig.1 Fig.1 shows a scetch of the phase potrait of the deterministic system in the vicinity of the marginal stable fixed point and the geometrical meaning of the coordinates (7). In the following discussion of the stochastic system I will concentrate on this bifurcation point of the deterministic motion.

2.1 The Gaussian approximation

The equations of motion for the stochastic system are given explicitely in Gaussian approximation by eqs.(2) in connection with eq.(4) and (5). For their analysis we use again the averaging procedure. Transforming the expectation values of position and velocity with eq.(6) and the fluctuations according to the linearized version, that means

$$\begin{pmatrix} \Delta\xi^2 \\ \Delta\xi\eta \\ \Delta\eta^2 \end{pmatrix} = \begin{pmatrix} \cos^2 t & -2\sin t\cos t & \sin^2 t \\ -\sin t\cos t & \sin^2 t - \cos^2 t & \sin t\cos t \\ \sin^2 t & 2\sin t\cos t & \cos^2 t \end{pmatrix} \begin{pmatrix} \Delta x^2 \\ \Delta xv \\ \Delta v^2 \end{pmatrix} \tag{9}$$

one gets a five dimensional system for slowly varying variables to which the averaging method can be applied. A little algebra yields

$$\begin{aligned}
\dot{\langle\xi\rangle} &= \delta\langle\eta\rangle - \langle\eta\rangle(\langle\xi\rangle^2 + \langle\eta\rangle^2) - \bar\gamma\langle\xi\rangle - \langle\eta\rangle\Delta\xi^2 - 2\langle\xi\rangle\Delta\xi\eta - 3\langle\eta\rangle\Delta\eta^2 \\
\dot{\langle\eta\rangle} &= -\delta\langle\xi\rangle + \langle\xi\rangle(\langle\xi\rangle^2 + \langle\eta\rangle^2) - \bar\gamma\langle\eta\rangle + h + \langle\xi\rangle\Delta\eta^2 + 2\langle\eta\rangle\Delta\xi\eta + 3\langle\xi\rangle\Delta\xi^2 \\
\dot{\Delta\xi^2} &= 2\{(\delta - \langle\xi\rangle^2 - 3\langle\eta\rangle^2)\Delta\xi\eta - (\bar\gamma + 2\langle\xi\rangle\langle\eta\rangle)\Delta\xi^2 \\
&\quad -3\Delta\xi\eta(\Delta\xi^2 + \Delta\eta^2) + \bar D\} \\
\dot{\Delta\xi\eta} &= -(\delta - 3\langle\xi\rangle^2 - \langle\eta\rangle^2)\Delta\xi^2 + (\delta - 3\langle\eta\rangle^2 - \langle\xi\rangle^2)\Delta\eta^2 - 2\bar\gamma\Delta\xi\eta \\
&\quad -3((\Delta\eta^2)^2 - (\Delta\xi^2)^2) \\
\dot{\Delta\eta^2} &= 2\{-(\delta - \langle\eta\rangle^2 - 3\langle\xi\rangle^2)\Delta\xi\eta - (\bar\gamma - 2\langle\xi\rangle\langle\eta\rangle)\Delta\eta^2 \\
&\quad +3\Delta\xi\eta(\Delta\xi^2 + \Delta\eta^2) + \bar D\} \quad .
\end{aligned} \tag{10}$$

where additional the new time scale τ has been introduced. These equations possess a similar structure as the original eqs.(2). Especially linearizing the deterministic part yields eqs.(10_{3-5}) except for quadratic terms in the fluctuations.

As mentioned above we will concentrate the discussion on the Codimension 2 bifurcation point $\delta = \delta_c, h = h_c$. In this case the steady state can be obtained from eq.(10) in the limit of small diffusion. But one must notice that a simple expansion in $\bar D$ around the deterministic fixed point $\langle\xi\rangle = \xi_0$, $\langle\eta\rangle = \eta_0$, $\Delta\xi^2 = \Delta\xi\eta = \Delta\eta^2 = 0$ fails. The coefficient matrix of the linearized system is singular and cannot be inverted as the parameter values correspond to a bifurcation point. Therefore the expectation values were significantly determined by the nonlinear terms in the fluctuations. The explicit calculation is usefully carried out in coordinates that are adapted to the deterministic center manifold (cf. eq.(7))

$$
\begin{aligned}
\langle \xi \rangle &=: \sqrt{\delta_c}\left(\frac{1}{\sqrt{6}} + \mu \frac{\langle \tilde{z} \rangle}{\sqrt{3}} + \mu \langle \tilde{y} \rangle\right) \\
\langle \eta \rangle &=: \sqrt{\delta_c}\left(\frac{1}{\sqrt{2}} - \mu \langle \tilde{z} \rangle\right) \\
\Delta\xi^2 &=: \delta_c\left(\mu \frac{\Delta\tilde{z}^2}{3} + \frac{2}{\sqrt{3}}\mu^2 \Delta\tilde{yz} + \mu^2 \Delta\tilde{y}^2\right) \\
\Delta\xi\eta &=: \delta_c\left(-\frac{1}{\sqrt{3}}\mu \Delta\tilde{z}^2 - \mu^2 \Delta\tilde{yz}\right) \\
\Delta\eta^2 &=: \delta_c \mu \Delta\tilde{z}^2 \quad .
\end{aligned}
\tag{11}
$$

We have used the abbreviation $\mu^2 := \frac{\bar{D}}{\delta_c^2}$ in these equations. The scaling of the new variables with this parameter is choosen in such a way that the resulting equations allow an iterative computation of the fixed point. This scaling must be determined appropriately after the computation of the transformed equations has been done. It is anticipated here for simplicity. With the transformation (11) eq.(10) reads

$$
\begin{aligned}
\frac{1}{\delta_c}\langle \dot{\tilde{z}} \rangle &= \mu\left(-\frac{3}{\sqrt{6}}\langle \tilde{y} \rangle^2 - \frac{3}{\sqrt{6}}\Delta\tilde{y}^2 - \frac{4}{\sqrt{3}}\langle \tilde{z} \rangle \Delta\tilde{z}^2 - 2\langle \tilde{y} \rangle \Delta\tilde{z}^2 + O(\mu)\right) \\
\frac{1}{\delta_c}\langle \dot{\tilde{y}} \rangle &= -\frac{2}{\sqrt{3}}\langle \tilde{y} \rangle - \frac{8}{3\sqrt{2}}\Delta\tilde{z}^2 + O(\mu) \\
\frac{1}{\delta_c}\dot{\Delta\tilde{z}^2} &= \mu\left(-\frac{8}{\sqrt{3}}(\Delta\tilde{z}^2)^2 + 2 + O(\mu)\right) \\
\frac{1}{\delta_c}\dot{\Delta\tilde{zy}} &= \frac{16}{3}(\Delta\tilde{z}^2)^2 - \frac{2}{\sqrt{3}} - \frac{2}{\sqrt{3}}\Delta\tilde{zy} - \frac{16}{3\sqrt{2}}\langle \tilde{z} \rangle \Delta\tilde{z}^2 + O(\mu) \\
\frac{1}{\delta_c}\dot{\Delta\tilde{y}^2} &= \frac{8}{3} - \frac{4}{\sqrt{3}}\Delta\tilde{y}^2 + O(\mu) \quad .
\end{aligned}
\tag{12}
$$

The fixed point can be obtained easily in the lowest nonvanishing order in μ.

$$
\langle \xi \rangle = \xi_0 - \mu\sqrt{\delta_c} \cdot 1.611\ldots \qquad \langle \eta \rangle = \eta_0 + \mu\sqrt{\delta_c} \cdot 0.930\ldots
$$

$$\Delta\xi^2 = \frac{1}{3}\mu\delta_c \cdot 0.658\ldots \qquad \Delta\xi\eta = -\frac{1}{\sqrt{3}}\mu\delta_c \cdot 0.658\ldots \qquad \Delta\eta^2 = \mu\delta_c \cdot 0.658\ldots \tag{13}$$

This result shows a non analytical dependence on the diffusion constant. The expectation values deviate from their deterministic values ξ_0, η_0 as a consequence of the coupling to the fluctuations in eq.($10_{1,2}$). This deviation comes from the non Gaussian deformation of the full distribution function (cf. chapter 3). In addition the values of the fluctuations remain finite but are enormously enlarged with respect to the diffusion constant. Tracing back to the fact that the fluctuations take values of order $O(1)$ in the scaled variables (11) one recognizes that the elliptic contour lines of the approximating Gaussian follows the form of the center manifold. The ratio of their axis amounts $\mu^{\frac{1}{2}} : \mu$ (cf. Fig.2). One might suggest that the full distribution function nestle against the center manifold too. This reasonable assumption will be tested in the next paragraph by analysing the full Fokker– Planck equation.

2.2 Perturbative treatment of the Fokker– Planck equation

The Fokker– Planck equation of the stochastic Duffing oscillator is given by eq.(1), (4) and (5). To compute the steady state of this weakly time dependent equation we use again a kind of averaging procedure. Initially one transforms the Fokker– Planck equation to the slowly varying coordinates ξ and η (cf. eq.(6)).

$$\frac{\partial \rho}{\partial t} = -\epsilon \mathbf{\Lambda}(t)\rho(\xi, \eta, t) \tag{14}$$

As the Fokker– Planck operator $\mathbf{\Lambda}(t)$ is a 2π periodic function and the distribution function slowly varying in time we obtain the expected form (Appendix A)

$$\frac{\partial \rho}{\partial \tau} = -\bar{\mathbf{\Lambda}}\rho(\xi, \eta, \tau) \qquad \bar{\mathbf{\Lambda}} = \frac{1}{\pi}\int_0^{2\pi} \mathbf{\Lambda}(t)dt \qquad \tau := \frac{\epsilon}{2}t \quad . \tag{15}$$

In the case regarded above the computations can easily be done and lead to [2]

$$\frac{\partial\rho}{\partial\tau} = \left(-\frac{\partial}{\partial\xi}D_\xi - \frac{\partial}{\partial\eta}D_\eta + \bar{D}\frac{\partial^2}{\partial\xi^2} + \bar{D}\frac{\partial^2}{\partial\eta^2}\right)\rho(\xi,\eta,\tau) \tag{16}$$

with the deterministic parts

$$\begin{aligned} D_\xi(\xi,\eta) &:= \delta\eta - \eta(\xi^2+\eta^2) - \bar{\gamma}\xi \\ D_\eta(\xi,\eta) &:= -\delta\xi + \xi(\xi^2+\eta^2) - \bar{\gamma}\eta + h \quad . \end{aligned} \tag{17}$$

As in the previous paragraph we again restrict the discussion to the case of the Codimension 2 bifurcation point $\delta = \delta_c, h = h_c$. The computation of the stationary state of eq.(16) can be carried out in the case of sufficiently small diffusion by perturbation methods which can be found in the literature. We will follow here the lines of [2] but refer closely to the geometrical structure of the deterministic phase space. It is appropriate to transform the Fokker–Planck equation to coordinates adapted to the deterministic center manifold (cf. eq.(7) and (8))

$$\begin{aligned} \xi &= \sqrt{\delta_c}\left(\frac{1}{\sqrt{6}} + \mu^{\frac{1}{2}}\frac{\hat{z}}{\sqrt{3}} + \mu\hat{y} - \frac{4}{\sqrt{6}}\mu\hat{z}^2\right) \\ \eta &= \sqrt{\delta_c}\left(\frac{1}{\sqrt{2}} - \mu^{\frac{1}{2}}\hat{z}\right) \quad . \end{aligned} \tag{18}$$

The scaling of the new variables with the diffusion constant is chosen in such a way that the transformed Fokker– Planck equation possesses drift and diffusion coefficients of the same order of magnitude. This structure allows a perturbative treatment of the stationary solution [3]. The transformed Fokker–Planck equation reads

$$\frac{1}{\delta_c}\frac{\partial\rho}{\partial\tau} = \left(-\frac{\partial}{\partial\hat{z}}D_{\hat{z}} - \frac{\partial}{\partial\hat{y}}D_{\hat{y}} + \frac{\partial^2}{\partial\hat{z}^2}D_{\hat{z}\hat{z}} + 2\frac{\partial^2}{\partial\hat{z}\partial\hat{y}}D_{\hat{z}\hat{y}} + \frac{\partial^2}{\partial\hat{y}^2}D_{\hat{y}\hat{y}}\right)\rho(\hat{z},\hat{y},\tau) \tag{19}$$

[2] The Gaussian approximation of this Fokker– Planck equation yields eq.(10).

[3] In mathematical terms one hopes to construct an asymptotic series for the solution.

with

$$D_{\hat{z}} = -\mu \frac{4}{3\sqrt{3}} \hat{z}^3 + O(\mu^{\frac{3}{2}}) \qquad D_{\hat{y}} = -\frac{2}{\sqrt{3}} \hat{y} + O(\mu^{\frac{1}{2}})$$

$$D_{\hat{z}\hat{z}} = \mu \quad D_{\hat{z}\hat{y}} = O(\mu^{\frac{1}{2}}) \qquad D_{\hat{y}\hat{y}} = \frac{4}{3} + O(\mu^{\frac{1}{2}}) \quad . \tag{20}$$

It has the property to separate in lowest order in μ. Inserting the definition of the reduced and conditional probability distribution of the stationary solution

$$P(\hat{z}) := \int \rho_{eq.}(\hat{z}, \hat{y}) d\hat{y} \qquad \rho_{eq.}(\hat{z}, \hat{y}) =: P(\hat{z}) W(\hat{y}|\hat{z}) \tag{21}$$

into eq.(19) integrating on one hand over $\hat{y}$ and dividing on the other hand by $P(\hat{z})$ we get in the lowest nonvanishing order in μ two ordinary differential equations for the functions P and W which can be integrated easily. One obtains

$$\begin{aligned} P(\hat{z}) &= \frac{1}{Z} \exp\left(-\frac{1}{3\sqrt{3}} \hat{z}^4 + O(\mu^{\frac{1}{2}})\right) \\ W(\hat{y}|\hat{z}) &= \frac{1}{Z} \exp\left(-\frac{3}{4\sqrt{3}} \hat{y}^2 + O(\mu^{\frac{1}{2}})\right) \quad . \end{aligned} \tag{22}$$

This result shows clearly that the distribution function is nearly Gaussian distributed transversal to the center manifold $\hat{y} = 0$. With respect to the scaling of the variables with the diffusion constant (18) one recognizes that the extension of the stationary distribution along the center manifold is en-
$\rightarrow$ Fig.2 larged by a factor $\mu^{\frac{1}{2}}$ (cf. Fig.2). As a consequence of this behaviour the expectation values deviate from their deterministic values. Especially $\langle \xi \rangle$ is smaller than ξ_0. Refering back to the fact that the deterministic forces are larger transversal to the center manifold as in its tangent direction the structure of the distribution function can also be intuitively understood. Furthermore the fluctuations take enlarged values with respect to the diffusion constant.

The result (22) allows finally the computation of the interesting expectation values. As obviously $\langle \hat{z} \rangle = O(\mu^{\frac{1}{2}})$, $\langle \hat{y} \rangle = O(\mu^{\frac{1}{2}})$ and $\langle \hat{z}\hat{y} \rangle = O(\mu^{\frac{1}{2}})$ holds one gets by taking expectation values in eq.(18) and in the squared expressions

$$
\begin{aligned}
\langle\xi\rangle &= \xi_0 + O(\mu) \\
\langle\eta\rangle &= \eta_0 + O(\mu) \\
\Delta\xi^2 &= \frac{1}{3}\mu\delta_c\Delta\hat{z}^2 + O(\mu^{\frac{3}{2}}) \\
\Delta\xi\eta &= -\frac{1}{\sqrt{3}}\mu\Delta\hat{z}^2 + O(\mu^{\frac{3}{2}}) \\
\Delta\eta^2 &= \mu\delta_c\Delta\hat{z}^2 + O(\mu^{\frac{3}{2}}) \quad ,
\end{aligned}
\tag{23}
$$

where

$$
\Delta\hat{z}^2 = \int \hat{z}^2 P(\hat{z})\, d\hat{z} = 0.770\ldots + O(\mu^{\frac{1}{2}}) \quad . \tag{24}
$$

Comparing this result with eq.(13) we obtain even a quantitative satisfactory agreement if one remembers that the full distribution function differs remarkably from a Gaussian structure [4].

2.3 The Cusp bifurcation

To complete the discussion of the weakly driven Duffing oscillator it would be useful to analyse a neighbourhood of the critical point (δ_c, h_c). This can be done in principle by using eqs.(10) and (16) and the standard procedures presented above. As this program involves tedious computations I resign this and treat for clearness a simple normal form that contains the main features and allows the analytical discussion of the stochastic system in Gaussian approximation.

It is well known [6, 7] that center manifold and normal form reduction for the dynamics of the deterministic Duffing oscillator (4) in the vicinity of the cusp bifurcation point leads to an equation of motion $\dot{z} = -U'(z)$ with the "potential"

$$
U(z) = -\beta z - \frac{\alpha}{2}z^2 + \frac{1}{4}z^4 \quad . \tag{25}
$$

[4] To obtain the $O(\mu)$ contributions in eq.$(23_{1,2})$ one has to compute the distribution (22) up to the order $O(\mu^{\frac{1}{2}})$.

Here the parameters α and β depend on $\delta - \delta_c$ and $h - h_c$ in a complicated manner. The motion in this potential shows the familiar Cusp scenario. We now assume that the dynamics of the stochastic Duffing oscillator can be described by inclusion of a stochastic force [5]. The Fokker– Planck equation reads

$$\frac{\partial \rho}{\partial t} = \left(\frac{\partial}{\partial z} U'(z) + \frac{\partial^2}{\partial z^2} \right) \rho(z,t) \quad . \tag{26}$$

This model system whose stationary solution can immeaditely be written down will be analysed in the subsequent paragraph to test the Gaussian approximation. Performing a cumulant expansion of eq.(26) up to the second order we get equations for the expectation value and the fluctuation [4].

$$\begin{aligned} \dot{\langle z \rangle} &= \beta + \alpha \langle z \rangle - \langle z \rangle^3 - 3\langle z \rangle \Delta z^2 \\ \dot{\Delta z^2} &= -2(3\langle z \rangle^2 + 3\Delta z^2 - \alpha)\Delta z^2 + 2 \end{aligned} \tag{27}$$

To construct the bifurcation diagramm of eq.(27) we analyse the coefficient matrix $\underline{\underline{C}}$ of the linearized system. As in our case $\mathrm{Sp}\,\underline{\underline{C}} < 0$ only the condition

$$0 = \mathrm{Det}\,\underline{\underline{C}} = 2(3\langle z \rangle^2 + 3\Delta z^2 - \alpha)(3\langle z \rangle^2 + 6\Delta z^2 - \alpha) - 36\langle z \rangle^2 \Delta z^2 \tag{28}$$

leads to a bifurcation (saddle node). Together with the equations determing the fixed points (cf. eq.(27)) one gets after some simple algebra

$$\begin{aligned} \langle z \rangle &= \pm \sqrt{\frac{1 + 3(\Delta z^2)^2}{18(\Delta z^2)^3}} & \Delta z^2 &\in [0, \infty) \\ \alpha &= \frac{18(\Delta z^2)^4 - 3(\Delta z^2)^2 + 1}{6(\Delta z^2)^3} & \beta &= \langle z \rangle \frac{6(\Delta z^2)^2 - 1}{9(\Delta z^2)^3} \quad . \end{aligned} \tag{29}$$

Eq.(29_2) yields two bifurcation lines in the parameter space (α, β) in parameter representation. The location $(\langle z \rangle, \Delta z^2)$ in the phase space where the
→ Fig.3 bifurcation occurs is given by eq.(29_1). Fig.3 shows the bifurcation lines, the number of fixed points (stable+ unstable) in the corresponding parameter region (cf. Appendix B) and the bifurcation line of the deterministic

[5] While in general the stochastic force may be multiplicative I have for simplicity assumed an additive stochastic force.

problem. For negative values of α and below the bifurcation lines only one stable fixed point exists which corresponds to a localized distribution in the minimum of the potential. One additional stable fixed point emerges if one crosses for sufficiently large values of $|\beta|$ the bifurcation lines into direction of increasing values of α. It corresponds to a localized distribution in the new potential minimum and is meta stable from the point of view of the full Fokker– Planck equation [4]. In the case $\beta \equiv 0$ the system reduces to the well known motion in a symmetric double well potential. For $\alpha < \sqrt{6}$ only one fixed point is present which corresponds to the stationary solution of the full stochastic system. Above this threshold two additional fixed points emerge which correspond to metastable distributions in one potential well [4]. If one crosses the bifurcation line into the direction of increasing values of $|\beta|$ for not to small values of α the fixed point corresponding to the bi-modal distribution function disappears. The situation in the vicinity of the origin is rather complex. It seems to be determined by the interaction of the stochastic force with the structure of the potential and can be reproduced only qualitatively by the Gaussian approximation. Apart from this small region of the parameter space the situation is described correctly.

In summarizing this section one may hope that the Gaussian approximation allows the treatment of complicated dynamical behaviour. As shown by the analytical computations and comparison with solutions of the Fokker–Planck equation the results are quite satisfactory even if the full distribution function deviates from the Gaussian structure.

3 The strongly driven Duffing oscillator

In this section we discuss the time evolution of a particle moving in a quartic potential and driven by a periodic force.

$$U(x,t) = \frac{1}{4}x^4 + xh\sin t \tag{30}$$

In contrast to the situation analysed in the preceding section the damping constant γ and the amplitude of the driving field h are not restricted to small values. For the following investigations we choose $\gamma = 0.3$ fixed and vary h

in the interval $[6.0, 7.8]$. To keep the discussion self contained I review first a few features of the deterministic dynamics [8] and then go on to investigate the stochastic system in detail.

3.1 The deterministic dynamics

To present the well known results of the numerical integration of the deterministic equations for the parameter values mentioned above one uses the Poincare technique [6] and chooses the cross section

$$\Sigma_{t_0=0} = \Big\{(x, v, t) \in \mathcal{R}^2 \times \mathcal{S}^1 | t = 0\Big\} \quad . \tag{31}$$

→ Fig.4 Fig.4 shows the familiar Feigenbaum scenario and a period four irregular attractor which emerges at $h = 7.8$.

In the following paragraph we will concentrate on the influence of the stochastic forces on this dynamical behaviour. I will investigate the modification of the Feigenbaum scenario in detail and only partially inquire into the influence on the irregular motion. Therefore I will not enter into the concepts of Liapunov exponents, generalized dimensions, etc. which have been often applied in the analysis of deterministic systems but cannot be used simply in the discussion of the stochastic counterpart.

3.2 The stochastic system

The discussion of the stochastic system (30) with the help of the full Fokker–Planck equation (1) seems to be a hopeless attempt. A numerical simulation of the corresponding stochastic differential equation demands on the other hand considerable numerical effort. Moreover the errors in the expectation values are in the order of magnitude of 10% as a consequence of the statistics in the ensemble averages. But the ordinary differential equations (2) derived in Gaussian approximation can be easily integrated numerically. They build a suitable starting point for the discussion of the bifurcation scenario in stochastic systems. In the following investigations we restrict the parameter values to $\gamma = 0.3$, $h \in [6.0, 7.8]$ and choose with $D = 0.3 \times 10^{-6}$ the diffusion constant so small that the quasi deterministic description by eqs.(2) seems to be possible. The computation of the steady states will lead us to the modification of the Feigenbaum scenario by stochastic forces. For clearness single regions of the parameter interval will be discussed separately. I use two dimensional projections of the five dimensional Poincare map to present the solutions.

α) *Period doubling*: The modification of the first period doubling emerging for $h \in [6.2, 6.5]$ is shown in Fig.5. One recognizes that the period one (P1) motion is stable along the whole interval in the stochastic case. The fluctuations are enhanced at the deterministic bifuration point. Closely above this threshold a period two (P2) motion emerges which corresponds to the deterministic dynamics (cf. small values of the fluctuations). This behaviour can be reproduced by the Gaussian approximation of simple stochastic mappings undergoing a flip bifurcation. Furthermore it shows strong resemblance to the stochastic pitchfork bifurcation discussed earlier [4]. In this sence the P2 motion corresponds to a localized metastable solution of the full Fokker–Planck equation while the P1 motion represents a double peak distribution whose extrema are located in the vicinity of the deterministic attractor.

$\rightarrow$ Fig.5

An analogous behaviour was found for the modification of the second period doubling at $h \sim 7.53$.

β) *Cascade of period doublings*: The influence of the stochastic force on the cascade emerging in the deterministic system in the interval $h \in [7.7, 7.8]$ is shown in Fig.6. No further period doubling can be established in the stochastic case. At $h = 7.765$ a four torus is born (probably into a Hopf bifurcation) which can clearly be seen by inspection of the reduced representation of the Poincare map in the $\langle x \rangle, \langle v \rangle$ plane (Fig.7). For parameter values $h \in [7.766, 7.768]$ the four torus exists. It changes for $h \rightarrow 7.769$ into a period 16 solution which consitsts of four period 4 branches. The limit cycle disappears in the Poincare map (Fig.7) and the solution locks at a commensurable frequency.

$\rightarrow$ Fig.6

$\rightarrow$ Fig.7

The corresponding scenario can be found for the P2 solution (cf. α)) which is also stable besides the P4 motion. It was suppressed in Fig.6 for clearness. These results can be collected schematically in the bifurcation diagramm presented by Fig.8. In contrast to the Feigenbaum scenario of

$\rightarrow$ Fig.8

the deterministic motion the stochastic system shows only a finite number of period doublings. This result is well known from numerical simulations of stochastic mappings [5] and shows striking similarity to results obtained from a moment expansion of the stochastic logistic equation [9].

So far we have not considered the physical meaning of the computed quantities. The structure of the bifurcation diagramm Fig.8 is of course only reasonable if it is not covered by the fluctuations. In the case of the period doublings this turns out to be true. But the situation changes considerably if one regards the torus solutions. Fig.9 shows the fluctuations in dependence on the expectation values for a particular value of h. Comparing this result with the extension of the limit cycle of the Poincare map (cf. Fig.7) one obtains that it is completly covered by the fluctuations. Therefore this type of the dynamics is not representative for the description of the stochastic system. But the result indicates a complicated dynamical behaviour in the full stochastic system which is characterized by the interaction of the fine structure of the deterministic attractor with the stochastic force and leads to the breakdown of the period doubling cascade.

$\rightarrow$ Fig.9

γ) *Irregular region at* $h \sim 7.8$: As indicated by the preceding investigations one may hope that the qualitative features of the stochastic system are reproduced correctly by the Gaussian approximation beyond the breakdown of the period doubling sequence. For that reason let us study the system in this complicated parameter region. As a first point one must notice that the P2 and P4 motions discussed above run into a two respectively four periodic irregular attractor close above $h = 7.769$. As a consequence these attractors coexist even at $h = 7.8$. Increasing the bifurcation parameter further the four periodic attractor gets unstable at $h = 7.82$ (Fig.10). This scenario resembles to the inversion of the period doubling discussed in α) and might represent a kind of stochastic band merging in the Feigenbaum attractor.

$\rightarrow$ Fig.10

δ) *Behaviour of the fluctuations on the irregular attractor*: At $h = 7.8$ the system posesses a four periodic irregular attractor as mentioned in γ). This motion corresponds to the deterministic behaviour (cf. α)) especially as the fluctuations connected with this motion remain small. To analyse the behaviour of the fluctuations let us surround every point in the $\langle x\rangle, \langle v\rangle$ phase plane visited by the solution with an ellipsis which shows a contour line of the approximating Gaussian (3). This representation connects the points of the phase space with its fluctuations and yields information about the approximating Gaussian. Additional the behaviour of the full distribution function can be guessed from that representation. The result of the numerical integration (Fig.11) shows that the distribution function tries to follow the form of the deterministic attractor. At a first sight this behaviour does not seem surprising since the hyperbolic structure of the deterministic attractor will lead to a delocalization of the distribution function in the unstable direction. On the other hand it is not trivial that this result can be reproduced by eqs.(2). One has to remark that the linear and the nonlinear terms in the fluctuations in eq.(2_{3-5}) lead to the observed behaviour. The first one build up the linearized deterministic equations and are responsible for the delocalization of the distribution function into the unstable direction. But the nonlinear terms have precisely the structure to confine the fluctuations to the deterministic attractor as it should be. So the situation is up to a certain degree similar to the foundings in simple Codimension 1 bifurcations [4].

→ Fig.11

ϵ) *Summary*: The discussion of the driven Duffing oscillator shows that the stochastic system can be analysed with the help of the Gaussian approximation. We have found the modification of a single period doubling by stochastic forces and the breakdown of the Feigenbaum cascade. As similar results can be obtained by numerical simulations of simple stochastic mappings I believe that these findings reflect quantitatively the dynamics of the stochastic system. Beyond the breakdown of the period doubling sequence one has to expect only qualitatively correct results. But also these solutions show remarkable features. One can observe that the distribution function follows the form of the irregular deterministic attractor. But as a consequence of the Gaussian approximation this behaviour is reproduced only imperfectly if the attractor deviates strongly from a straight line. Furthermore the highest period of the observed regular motion which is followed by a complicated dynamics coincides with the period of the irregular attractor emerging at higher parameter values. Both kind of motions are separated by a "gap" as was mentioned earlier in connection with the stochastic logistic equation [5].

4 Conclusion

The analytical and numerical results presented above show that even complicated dynamical behaviour of stochastic systems can be described by ordinary differential equations in the framework of the Gaussian approximation. These equations can be handled with moderate numerical effort in contrast to the full stochastic description. It was therefore possible to discuss the modification of the Feigenbaum scenario emerging in the Duffing oscillator by a stochastic force. Whenever the diffusion constant is sufficiently small and the distribution function remains localized the method presented seems to be applicable. For practical purposes it will be desireable to reduce the dimension of the system of differential equations. This might be possible in a special context by eliminating irrelevant variables.

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Appendix A

The problem of averaging in Fokker– Planck equations can be found in textbooks [10]. But in most cases the calculations were done on physical arguments. To my surprise I could not find a formal expansion in the small parameter ϵ in the literature. I therefore adapt an idea which was used in a different context [11] to achieve this goal [6].

[6] As the operators involved are not bounded the following manipulations have only a formal meaning.

One uses the Floquet decomposition of the evolution operator $\rho(t) = \mathbf{U}(t)\rho(0)$

$$\mathbf{U}(t) = \mathbf{Q}(t)e^{-\mathbf{C}t} \qquad \mathbf{Q}(t) = \mathbf{Q}(t+2\pi) \tag{32}$$

to define a transformed (not necessarily positive) distribution function $\bar{\rho}(t) := \mathbf{Q^{-1}}(t)\rho(t)$. Then eq.(14) yields

$$\frac{\partial \bar{\rho}}{\partial t} = -\mathbf{C}\bar{\rho}(t) \tag{33}$$

which is the averaged version of the original Fokker– Planck equation. The special structure of eq.(14) allows a perturbative computation of the operator $\mathbf{C}$ and of the transformation $\mathbf{Q}(t)$. Insertion of eq.(32) into the equation of motion $\dot{\mathbf{U}} = -\epsilon\mathbf{\Lambda}(t)\mathbf{U}(t)$ leads to

$$\dot{\mathbf{Q}}(t) - \mathbf{Q}(t)\mathbf{C} = -\epsilon\mathbf{\Lambda}(t)\mathbf{Q}(t) \quad . \tag{34}$$

Using the formal expansions

$$\begin{aligned} \mathbf{Q}(t) &= \mathbf{1} + \sum_{\nu=1}^{\infty} \epsilon^{\nu}\mathbf{Q}_{\nu}(t) \\ \mathbf{C} &= \sum_{\nu=1}^{\infty} \epsilon^{\nu}\mathbf{C}_{\nu} \end{aligned} \tag{35}$$

one gets by comparison of equal powers in ϵ in reference to the periodicity of $\mathbf{Q}_{\nu}(t)$ [7]

$$\begin{aligned} \mathbf{C}_{\nu} &= \frac{1}{2\pi}\int_0^{2\pi} \mathbf{\Lambda}(\tau)\mathbf{Q}_{\nu-\mathbf{1}}(\tau)\,d\tau - \sum_{\mu=1}^{\nu-1}\frac{1}{2\pi}\int_0^{2\pi}\mathbf{Q}_{\nu-\mu}(\tau)\,d\tau\,\mathbf{C}_{\mu} \\ \mathbf{Q}_{\nu}(t) &= \mathbf{C}_{\nu}t - \int_0^{t}\mathbf{\Lambda}(\tau)\mathbf{Q}_{\nu-\mathbf{1}}(\tau)\,d\tau + \sum_{\mu=1}^{\nu-1}\int_0^{t}\mathbf{Q}_{\nu-\mu}(\tau)\,d\tau\,\mathbf{C}_{\mu} \quad . \end{aligned} \tag{36}$$

The expansion (35_1) shows that the transformation to the new distribution function is to a certain extent a near identity transformation. So as usual in averaging the bar in the notation will be suppressed if one is interested only

[7] $\mathbf{Q_0}(t) := \mathbf{1}$

in the lowest nonvanishing order. In this case one has $\mathbf{C} = \epsilon \mathbf{C_1}$ and gets the result (15) which has only in this order Fokker– Planck form again. Finally let me still mention that by this method the ordinary averaging procedure will be recovered in the deterministic case.

Appendix B

Let $\alpha_0, \beta_0, \langle z\rangle_0, \Delta z_0^2$ satisfy eq.(29) [8] that means α_0, β_0 are parameter values and $\langle z\rangle_0, \Delta z_0^2$ a point in the phase space at which a bifurcation occurs. To decide on which side of the bifurcation line the saddle node pair emerges let us choose a neighbourhood of parameter values $\alpha = \alpha_0$, $\beta = \beta_0 + \lambda^2\beta_1$ (λ expansion parameter) and write down the formal expansions of the fixed point values

$$\begin{aligned} \langle z\rangle &= \langle z\rangle_0 + \lambda\langle z\rangle_1 + \lambda^2\langle z\rangle_2 + \cdots \\ \Delta z^2 &= \Delta z_0^2 + \lambda\Delta z_1^2 + \lambda^2\Delta z_2^2 + \cdots \quad . \end{aligned} \tag{37}$$

Inserting these into eq.(27) one gets

$$0 = \underline{\underline{C}}_0 \begin{pmatrix} \lambda\langle z\rangle_1 + \lambda^2\langle z\rangle_2 \\ \lambda\Delta z_1^2 + \lambda^2\Delta z_2^2 \end{pmatrix} - \lambda^2 \begin{pmatrix} -\beta_1 + 3\langle z\rangle_0\langle z\rangle_1^2 + 3\langle z\rangle_1\Delta z_1^2 \\ 6\langle z\rangle_1^2\Delta z_0^2 + 12\langle z\rangle_0\langle z\rangle_1\Delta z_1^2 + 6(\Delta z_1^2)^2 \end{pmatrix} + O(\lambda^3) \tag{38}$$

where

$$\underline{\underline{C}}_0 = \begin{pmatrix} -\frac{1}{\Delta z_0^2} & -3\langle z\rangle_0 \\ -12\langle z\rangle_0\Delta z_0^2 & -\frac{2}{\Delta z_0^2} - 6\Delta z_0^2 \end{pmatrix} \tag{39}$$

denotes the coefficient matrix of the linearized system. Equating equal powers in λ one obtains that the lowest order is proportional to the right null eigenvector of $\underline{\underline{C_0}}$

$$\begin{pmatrix} \langle z\rangle_1 \\ \Delta z_1^2 \end{pmatrix} = c \begin{pmatrix} -3\langle z\rangle_0 \\ \frac{1}{\Delta z_0^2} \end{pmatrix} \quad . \tag{40}$$

[8] In the following I assume the + sign in eq.(29). The opposite case can be treated in the same way.

Multiplying eq.(38) with the left null eigenvector of $\underline{\underline{C_0}}$ and inserting eq.(40) we get a relation by which the unknown quantity c is determined

$$0 = 12\beta_1 \langle z \rangle_0 \Delta z_0^2 - c^2(1 - 3(\Delta z_0^2)^2) \left(\frac{18\langle z \rangle_0^2}{(\Delta z_0^2)^2} + \frac{2 + 3(\Delta z_0^2)^2}{(\Delta z_0^2)^5} \right) \quad . \qquad (41)$$

In the case $\Delta z_0^2 < \frac{1}{\sqrt{3}}$ ($\Delta z_0^2 > \frac{1}{\sqrt{3}}$) this equation has a solution for $\beta_1 > 0$ ($\beta_1 < 0$). As the vertex of the bifurcation line is given by $\Delta z_0^2 = \frac{1}{\sqrt{3}}$ one recognizes that the saddle node pair emerges if the bifurcation line is crossed into the direction of increasing (decreasing) values of β.

Figure captions

Fig.1: Center (C.M.) and stable manifold (S.M.) into a neighbourhood of the marginal stable fixed point (ξ_0, η_0). Additionally the geometrical meaning of the coordinates z, y is shown.

Fig.2: Diagrammatic view of the stationary distribution (22) ( ) and of the approximating Gaussian ( ) determined by eq.(13). The figure shows also the deterministic center (C.M.) and stable manifold (S.M.).

Fig.3: Bifurcation diagramm of eq.(27). Besides the number of stable and unstable fixed points (s+u) the figure contains at the bifurcation lines the values of Δz^2 at which the bifurcation occurs. ( ) represents the bifurcation line of the deterministic problem.

Fig.4: a) Feigenbaum scenario of the deterministic equations in the variable x. The period doublings emerge at $h = 6.3,\ 7.5,\ 7.72,\ 7.77\ldots$.
b) Attractor of the Poincare map for $h = 7.8$.

Fig.5: Bifurcation diagramms in the deterministic (a) $D = 0$) and stochastic case (b) $D = 0.3 \times 10^{-6}$) for $h \in [6.2, 6.5]$. PM_n denotes the period M of the solution and n the sequence in which the branches are passed. Analogous representations follow for the dependence of $\langle v \rangle$, Δvx and Δv^2 on h.

Fig.6: Bifurcation diagramms in the deterministic (a) $D = 0$) and stochastic case (b) $D = 0.3 \times 10^{-6}$) for $h \in [7.7, 7.8]$. PM_n denotes the period M of the solution and n the sequence in which the branches are passed. Analogous representations follow for the dependence of $\langle v \rangle$, Δvx and Δv^2 on h.

Fig.7: Attractor of the Poincare map projected onto the $\langle x \rangle$, $\langle v \rangle$ plane for different values of the parameter h ($D = 0.3 \times 10^{-6}$). The figures show only

a part of the four periodic attractor.

Fig.8: Diagrammatic view of the bifurcation diagramm (L.C. limit cycle respectively torus solution, S.C. homoclinic bifurcation). For clearness only a few branches are shown explicitly.

Fig.9: Fluctuations in dependence on the expectation values for $h = 7.766$ ($D = 0.3 \times 10^{-6}$) (cf. Fig.7).

Fig.10: Separate representations of the coexisting P2 (b)) and P4 attractors (a)) of the Poincare map for two values of h ($D = 0.3 \times 10^{-6}$). The figure shows only a part of the phase plane.

Fig.11: a) Contour lines of the approximating Gaussian for $h = 7.8$, $D = 0.3 \times 10^{-6}$. The figure shows only a part of the phase plane and for clearness only four data sets. The contour lines are taken at the maximum probability divided by Eulers constant.
b) Corresponding part of the deterministic attractor.

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