Globally Coupled Maps: Phase Transitions and Synchronization

Wolfram Just
Theoretische Festkörperphysik
Technische Hochschule Darmstadt
Hochschulstraße 8
D–64289 Darmstadt
Germany

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Abstract

Bifurcations in a system of coupled maps are investigated. Using symbolic dynamics it is shown that for coupled shift maps the well known space–time–mixing attractor becomes unstable at a critical coupling strength in favour of a synchronized state. For coupled non–hyperbolic maps analytical and numerical evidence is given that arbitrary small coupling changes the dynamical behaviour. The anomalous dependence of fluctuations on the system size is attributed to these bifurcations.

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1 Introduction

The influence of chaotic dynamics in spatially extended systems is a field of intensive research. The competition between local chaos and diffusive coupling seems to be at the heart of pattern formation out of a random state. A detailed knowledge of this mechanism seems to be necessary to understand the development of structures out of regular and irregular states (cf. [1] and references cited therein). These concepts link different fields like e.g. hydrodynamic, optical and magnetic instabilities, chemical reactions or biological systems. It is usually tremendous complicated to gain information from the basic equations of motion even if numerical simulations are concerned. For that reason one is forced to investigate simple model systems, a strategy which has proven to be fruitful in several fields of low dimensional chaotic systems [2, 3]. One class of suitable model systems are constituted by coupled maps which are known to incorporate many features of real time evolution from a phenomenological point of view (e.g. [4]). They are accessible not only by numerical methods with moderate effort but also allow for a partially rigorous treatment. Especially the weak coupling limit of hyperbolic coupled maps has been considered recently [5, 6]. It has been shown that even in the limit of infinite system size the dynamics is dominated by exponentially decaying correlation functions, a property which is usually termed space–time–mixing. The motion is similar to the dynamics of the uncoupled system. Indeed it has been shown that the map lattice can be decoupled by a continuous transformation. Furthermore it has been suggested that pattern formation can be related to equilibrium phase transitions in higher dimensional spin systems using a thermodynamical formulation via symbolic dynamics. However no definite results are available at the moment. Furthermore (partially) solvable model systems are lacking which are suitable to study these problems.

It is the aim of this paper to investigate a simple coupled map lattice beyond the weak coupling regime. To be definite a lattice of maps on the circle $S^1 = [0, 2\pi]$ is considered. The circle and not the interval is chosen as phase space because hyperbolicity, the prerequisite to obtain rigorous results, is guaranteed for local expanding maps [7]. The equations of motion read

$$\varphi_{n+1}^{(\nu)} = \left(T(\underline{\varphi}_n)\right)^{(\nu)} = f(\varphi_n^{(\nu)}) + \frac{\epsilon}{L}\sum_{\mu} g\left(f(\varphi_n^{(\mu)}) - f(\varphi_n^{(\nu)})\right) \;|\, \mathrm{mod}\, 2\pi \quad . \tag{1}$$

L denotes the lattice size, ϵ the strength of the coupling, and $\underline{\varphi} = (\varphi^{(\nu)}) \in S^L$ the phase space coordinates. The map f which determines the single site dynamics is supposed to be given by

$$f(\varphi) = 2\varphi + a\sin(\varphi) \;|\, \mathrm{mod}\, 2\pi \quad . \tag{2}$$

For different values of the parameter $a \in [0, 2]$ it interpolates between the simple case of the shift map ($a = 0$) and a strongly non–hyperbolic map which has a critical point ($a = 2$). In the latter case it resembles strongly the structure of the Smale complete logistic equation. Continuity of the map T requires periodicity for the coupling function g. For explicit calculations the simplest choice

$$g(x) = \sin(x) \tag{3}$$

will be considered. A global coupling between the different lattice sites has been assumed in eq.(1). This type of coupling considerably simplifies a theoretical approach. It has also been suggested that such a mean field approach yields the correct description of physically more reasonable models with short range coupling above some critical dimension in accordance to equilibrium statistical mechanics [4]. Furthermore globally coupled models are interesting in their own right. They may be considered on one hand as limiting cases of models with long range interactions [8]. On the other hand they have been considered in the context of biology, neural networks and Hamiltonian dynamics of coupled oscillators [9, 10, 11, 12]. Special emphasis has been laid on the mechanism of synchronization between the different elements. Analogous phenomena can be studied in the model (1). In contrast to the mentioned approaches the randomness is not modeled by a stochastic force but is an inherent property of the dynamical system.

The plan of the paper reads as follows. Section 2 reviews the well known construction of a symbolic dynamics for the coupled map lattice (1) on an elementary level. With its help a space–time–mixing stationary state can be established for sufficiently small coupling strength $\epsilon < \epsilon_c^e$. In the subsequent sections the two limiting cases of hyperbolic coupled maps ($a = 0$) and strongly non–hyperbolic coupled maps ($a \lesssim 2$) are discussed separately. In section 3 it is shown that the breakdown of the space–time–mixing regime is accompanied by a global synchronization among the lattice sites for moderate coupling strength $\epsilon_c^e < \epsilon < \epsilon_c^m$. Furthermore space–time–chaotic transients of tremendous length are observed which are attributed to a hyperbolic repeller. The case of non–hyperbolic coupled maps is much more difficult to analyse. In section 4 analytical and numerical evidence is given that even for infinitesimal coupling strength complicated bifurcations arise. They cause an anomalous dependence of the fluctuations on the system size, an effect which has been commonly observed in globally coupled non–hyperbolic maps and has been termed "violation of the law of large numbers" [13, 14, 15, 16]. Finally comments on prospective work are given.

2 Symbolic dynamics for coupled map lattices

It is the objection of this paragraph to review the well known results on the symbolic dynamics of system (1) on an elementary level [5]. Additionally I will set up some notation which will be useful in the subsequent discussion.

To construct the symbolic dynamics only some global features of the single site map f and the coupling g is needed but not the explicit expressions (2) and (3). Especially one demands that the single site map is monotonous. It ensures that f admits a Markov partition[1] [7]. For simplicity in notation let us assume a binary partition which is written as $I_0 = [0, \pi]$, $I_1 = [\pi, 2\pi]$ without loss of generality. f maps these intervals to the whole phase space in a monotonous way, $f(I_i) = [0, 2\pi]$. With this partition a symbol sequence $(\sigma_0, \sigma_1, \ldots)$, $\sigma_i \in \{0, 1\}$ is assigned to every phase space point φ. It denotes the element I_{σ_k} of the partition that contains the image point $f^k(\varphi)$, $k \geq 0$ [2].

The Markov partition of the uncoupled map lattice, $\epsilon = 0$ can be obtained trivially as a direct product $U_{\underline{\sigma}} = I_{\sigma^{(0)}} \otimes \cdots \otimes I_{\sigma^{(L-1)}}$, $\underline{\sigma} = (\sigma^{(0)}, \ldots, \sigma^{(L-1)})$. The boundaries of these sets are given by the "hyperplanes" $\varphi^{(\nu)} = 0$ and $\varphi^{(\nu)} = \pi$ respectively. The elements of this partition are mapped by T to the whole phase space in a bijective way. If the single site map is expanding then the partition is also generating so that the partitions generated via $U^{(n)}_{\underline{\sigma}_0, \ldots, \underline{\sigma}_n} := \bigcap_{k=0}^{n} T^{-k}(U_{\underline{\sigma}_k})$ become finer for increasing n. Any sequence $\underline{\sigma}_0, \underline{\sigma}_1, \ldots$ specifies a phase space point and the dynamics is equivalent to a shift operation on the two dimensional spin lattice $\underline{\sigma}_0, \underline{\sigma}_1, \ldots$.

For the case of finite coupling one can also construct a Markov partition (cf. Appendix A). However I would like to use a slightly modified approach to construct a symbolic dynamics. The discussion is somewhat simplified if one considers the extended map $tT : \mathbb{R}^L \to \mathbb{R}^L$ which develops from the original evolution equation (1) by suppressing the modulo operation

$$x^{(\nu)}_{n+1} = \left(\tilde{T}(\underline{x}_n)\right)^{(\nu)} = \tilde{f}(x^{(\nu)}_n) + \frac{\epsilon}{L} \sum_{\mu} g\left(\tilde{f}(x^{(\mu)}_n) - \tilde{f}(x^{(\nu)}_n)\right) \quad . \tag{4}$$

Although this system has orbits which tend to infinity the original dynamics is recovered via $\varphi^{(\nu)}_n = x^{(\nu)}_n | \mathrm{mod} 2\pi$. Furthermore any object in the original phase space S^L (e.g. the partition $U_{\underline{\sigma}}$) can be identified with the corresponding object in the extended space $\mathbb{R}^L$ by 2π periodic continuation. The action of eq.(1) respectively (4) is locally invertible for moderate coupling. Using the linearization

$$\left(D\tilde{T}(\underline{x})\right)_{\nu\mu} = \tilde{f}'(x^{(\mu)}) \left[\delta_{\nu\mu} - \frac{\epsilon}{L} \sum_{\rho} g'(\tilde{f}(x^{(\rho)}) - \tilde{f}(x^{(\nu)}))\delta_{\nu\mu} + \frac{\epsilon}{L} g'(\tilde{f}(x^{(\mu)}) - \tilde{f}(x^{(\nu)}))\right] \tag{5}$$

one obtains for the determinant (cf. appendix B)

$$\det(D\tilde{T}(\underline{x})) = \prod_{\nu=0}^{L-1} \tilde{f}'(x^{(\nu)}) \left[\prod_{\nu} \left(1 - \frac{\epsilon}{L} \sum_{\mu} \gamma_{\nu\mu}\right) + \frac{\epsilon}{L} \sum_{\nu} \prod_{\rho(\neq\nu)} \left(1 - \frac{\epsilon}{L} \sum_{\mu} \gamma_{\rho\mu}\right)\right.$$

[1] If additionally f is locally expanding, $|f'| > 1$ then the partition is generating.

[2] More precisely: $\varphi \in \bigcap_{k \geq 0} f^{-k}(I_{\sigma_k})$ yields a homomorphism between the dynamics and the shift of symbol sequences.

$$+\frac{1}{2}\left(\frac{\epsilon}{L}\right)^2 \sum_{\nu\neq\rho}\left(1-\gamma_{\nu\rho}^2\right) \prod_{\mu(\neq\nu,\rho)}\left(1-\frac{\epsilon}{L}\sum_{\sigma}\gamma_{\mu\sigma}\right)\Bigg] \tag{6}$$

where the abbreviation $\gamma_{\nu\mu} = g'(\tilde{f}(x^{(\mu)}) - \tilde{f}(x^{(\nu)}))$ has been used. As $\gamma_{\nu\mu} \leq 1$ and $\tilde{f}' > 0$ by definition it follows that

$$\det(D\tilde{T}(\underline{x})) \geq \prod_{\nu=0}^{L-1} \tilde{f}'(x^{(\nu)}) \left[(1-\epsilon)^L + \epsilon(1-\epsilon)^{L-1} + \frac{1}{2}\epsilon^2 \frac{L-1}{L}(1-\epsilon)^{L-2}\right] \quad . \tag{7}$$

Hence the determinant does not vanish for $0 \leq \epsilon < \epsilon_c^m = 1$ which proves that $\tilde{T}$ can be inverted locally. In contrast to the map (1) the extended system (4) is in addition globally invertible because its phase space is simply connected. This property yields the main difference between both formulations.

In order to construct a symbolic dynamics it is useful to introduce a sequence of increasingly finer partitions. For that reason consider the coordinate grid $x^{(\nu)} = 2k^{(\nu)}\pi, k^{(\nu)} \in \mathbb{Z}$ in the extended phase space. Its first counterimage with respect to the map $\tilde{T}$ leads to a 2π periodic grid (cf. Fig. 1). The codimension one manifolds which determine this grid are determined by the equations < Fig.1

$$\begin{aligned} 0 &= H_{x^{(\mu)},\mu\neq\nu}(x^{(\nu)}) := \tilde{f}(x^{(\nu)}) + \frac{\epsilon}{L}\sum_{\mu} g\left(\tilde{f}(x^{(\mu)}) - \tilde{f}(x^{(\nu)})\right) \\ 2\pi &= H_{x^{(\mu)},\mu\neq\nu}(x^{(\nu)}) \quad . \end{aligned} \tag{8}$$

They possess a unique solution because $H'_{x^{(\mu)},\mu\neq\nu}(x^{(\nu)}) > 0$ holds for $0 \leq \epsilon < \epsilon_c^m$. The grid can be factored down to the original phase space S^L. On this space one obtains a partition in 2^N sets $U^{(0)}_{\underline{\sigma}_0}$. By construction the map $T : U^{(0)}_{\underline{\sigma}_0} \to S^L$ is invertible[3]. One now proceeds and considers the counterimage of $U^{(0)}_{\underline{\sigma}_1}$. It consists of 2^N connected components which can be labeled by an index $\underline{\sigma}_0$. In this way a different partition $U^{(1)}_{\underline{\sigma}_0,\underline{\sigma}_1}$ is constructed which fulfills the property that $T : U^{(1)}_{\underline{\sigma}_0,\underline{\sigma}_1} \to U^{(0}_{\underline{\sigma}_1}$ is invertible. If one continues this construction one ends up with a sequence of partitons $U^{(n)}_{\underline{\sigma}_0,\ldots,\underline{\sigma}_n}$ and a bijection $T : U^{(n)}_{\underline{\sigma}_0,\ldots,\underline{\sigma}_n} \to U^{(n-1)}_{\underline{\sigma}_0,\ldots,\underline{\sigma}_{n-1}}$. The indices can be arranged in such a way that the elements of the partition obey the intersection property $\bigcup_{\underline{\sigma}_n} U^{(n)}_{\underline{\sigma}_0,\ldots,\underline{\sigma}_n} \cap U^{(n-1)}_{\underline{\sigma}_0,\ldots,\underline{\sigma}_{n-1}} \neq \emptyset$ (cf. appendix C)[4].

The symbolic dynamics is now defined by "taking the limit $n \to \infty$". To be more precise one considers for fixed symbol sequence $\underline{\sigma}_0, \underline{\sigma}_1, \ldots$ the set $V_{\underline{\sigma}_0,\underline{\sigma}_1,\ldots} := \bigcap_{n\geq 0} \overline{\bigcup_{k\geq n} U^{(k)}_{\underline{\sigma}_0,\ldots,\underline{\sigma}_k}}$. It obeys $T(V_{\underline{\sigma}_0,\underline{\sigma}_1,\ldots}) = V_{\underline{\sigma}_1,\ldots}$. To make the partition a generating one, that means that the set $V_{\underline{\sigma}_0,\underline{\sigma}_1,\ldots}$ contains exactly one point, one has to impose

[3]Hence any phase space point has 2^N counterimages and the counterimage of every connected subset consists of 2^N connected components.

[4]For Markov partitions the sets on the left hand side coincide.

some expansion property. In such a case the diameters of the successive counterimages $U^{(n)}_{\underline{\sigma}_0,\cdots\underline{\sigma}_n}$ decrease exponentially and the intersection property guarantees that the sets $V_{\underline{\sigma}_0,\underline{\sigma}_1,\ldots}$ are disjoint and contain only one phase space point. The map T is obviously expansive if the map $\tilde{T}$ expands any points[5]. Straightforward calculation shows that this global expansion property coincides with local expansiveness. It states that infinitesimal neighbouring points separate in the course of the dynamics

$$\forall \underline{y} \quad \|D\tilde{T}(\underline{x})\underline{y}\| \geq c\|\underline{y}\|, \quad c > 1 \quad . \tag{9}$$

Expansiveness may pose additional constraints on the coupling strength ϵ. In the uncoupled case this property is obviously fulfilled for expanding maps, $|f'| > 1$. Hence it persists up to some critical value ϵ_c^e. In the range $0 \leq \epsilon < \epsilon_c^e$ the dynamics is equivalent to a shift in a two dimensional spin lattice $(\underline{\sigma}_0, \underline{\sigma}_1, \ldots)$. It proves that the coupled system can be decoupled by a continuous coordinate transformation[6]. The dynamics in this state is mixing with respect to time as well as space translations [5]. It is however difficult to compute the explicit value ϵ_c^e from eq.(9) without specifying the single site map and the coupling. So we postpone the evaluation to the next sections.

Let me close this paragraph by stressing briefly the opposite case of large coupling. Because of the infinite range of interaction any lattice sites which take the same phase space value, $\varphi_n^{(\nu)} = \varphi_n^{(\mu)}$, will maintain this property forever. Such a region of constant phase may be called a domain. The most trivial domain is the globally synchronized solution $\varphi_n^{(\nu)} = \phi_n$. The dynamics of such a solution is governed by the single site map $\phi_{n+1} = f(\phi_n)$. Its stability is determined by the linearized dynamics which in view of eq.(1) reads

$$\delta\varphi_{n+1}^{(\nu)} = f'(\phi_n)\left[(1 - \epsilon g'(0))\delta\varphi_n^{(\nu)} + \frac{\epsilon}{L} g'(0) \sum_\mu \delta\varphi_n^{(\mu)}\right] \quad . \tag{10}$$

The solution can be written as

$$\delta\varphi_n^{(\nu)} = \exp\left(i\frac{2\pi}{L}k\nu\right)\delta a_n(k), \quad 0 \leq k \leq L-1 \tag{11}$$

where the amplitudes obey

$$\delta a_{n+1}(k) = \lambda_k f'(\phi_n)\delta a_n(k), \quad \lambda_{k=0} = 1, \quad \lambda_{k\neq 0} = 1 - \epsilon g'(0) \quad . \tag{12}$$

Stability requires that the coefficient on the right hand side is smaller than unity on the average for $k \neq 0$. This condition determines a critical coupling strength ϵ_c^s via

$$\exp\left(\langle \ln|f'(\varphi)|\rangle\right)(1 - \epsilon_c^s g'(0)) = 1 \tag{13}$$

[5] more precisely: the image of any δ neighbourhood of a point $\underline{x}$ contains a δ' neighbourhood of $\tilde{T}(\underline{x})$ with $\delta' > \delta$.

[6] In general this transformation seems to be of no practical use because of its complicated structure.

where $\langle \cdots \rangle$ denotes the long time average. Synchronization sets in for $\epsilon > \epsilon_c^s$. Obviously $\epsilon_c^e \leq \epsilon_c^s$ holds. The transition from the space–time–mixing regime to the synchronized state will be at the center of interest of the next section.

3 Hyperbolic coupled maps

The most simple case of coupled shift maps, $a = 0$ has been discussed in the literature from several point of views (e.g. for mathematical considerations [5], for the case $L = 2$ [17]). It allows for a partially analytical treatment for moderate coupling $0 < \epsilon < \epsilon_c^m$ which incorporates the transition to the synchronized state. The discussion of this transition is at the heart of this section.

First of all the condition of expansivity eq.(9) can be evaluated easily. Using eqs.(2), (3) and (5) for $a = 0$ one obtains for arbitrary vectors $\underline{y}$ in the tangent space

$$\begin{aligned} \langle \underline{y} | D\tilde{T}(\underline{x}) | \underline{y} \rangle &= 2 \sum_\nu \left(y^{(\nu)} \right)^2 - \frac{2\epsilon}{L} \sum_{\nu,\rho} \cos(2x^{(\rho)} - 2x^{(\nu)}) \left(y^{(\nu)} \right)^2 \\ &\quad + \frac{2\epsilon}{L} \left| \sum_\mu \exp(i2x^{(\mu)}) y^{(\mu)} \right|^2 \\ &\geq (2 - 2\epsilon) \langle \underline{y} | \underline{y} \rangle \quad . \end{aligned} \tag{14}$$

Owing to the fact that the matrix (5) is symmetric for $a = 0$ the condition (9) follows from eq.(14) for $0 \leq \epsilon < \epsilon_c^e = 1/2$. It implies that the dynamics is equivalent to a shift operation in a two dimensional spin lattice $\underline{\sigma}_0, \underline{\sigma}_1, \ldots$ and is mixing with respect to the time evolution as well as space translations[7].

On the other hand the stability condition for the synchronized state (13) immediately yields a local stable uniform solution for $\epsilon_c^s = 1/2 < \epsilon < 3/2$. Hence the breakdown of space–time–mixing is related to a global synchronization in the system. It is the objective to analyse this transition from the space–time–mixing regime to the synchronized state.

Before we enter the discussion of the general case let me first consider the simple situation of two coupled shift maps. It allows for a complete analysis and contains the main ideas of the subsequent analysis. If one uses the extended description (4) and introduces symmetric coordinates $x^{(\pm)} = x^{(1)} \pm x^{(0)}$ the system decouples

$$\begin{aligned} x_{n+1}^{(+)} &= 2x_n^{(+)} \\ x_{n+1}^{(-)} &= 2x_n^{(-)} - \epsilon \sin\left(2x_n^{(-)}\right) \quad . \end{aligned} \tag{15}$$

Eqs. (15) can be considered modulo 4π because the variables $x^{(\nu)}$ are of interest only modulo 2π.

[7] On a rigorous mathematical level the latter statement requires additional investigations [5].

The first equation describes the chaotic motion parallel to the diagonal of the phase space. The dynamics perpendicular to this direction is governed by the second one. The corresponding map is depicted in Fig.2. For $0 \leq \epsilon < \epsilon_c^s$ the map possesses a slope larger than unity and yields a chaotic attracting set. Approaching ϵ_c^s from below intermittency near the fixed point $x^{(-)} = 0$ sets in. Above the critical value the fixed point becomes stable via a pitchfork bifurcation. Furthermore there remains an invariant repelling Cantor set in the region bounded by the two unstable fixed points. The transition is usually termed a boundary crisis [18]. The fixed point is globally stable but chaotic transients occur. < Fig.2

Let me try to treat the dynamics of the general system (4) in a quite similar fashion for $\epsilon_c^s < \epsilon < \epsilon_c^m$. To this end consider a neighbourhood of the diagonal

$$\tilde{A}_\delta = \{\underline{x} \in \mathbb{R}^L \,|\, |x^{(\nu)} - x^{(\mu)}| \leq \delta\} \quad . \tag{16}$$

First of all the diameter δ is chosen in such a way that the system (4) contracts the neighbourhood towards the synchronized state. For $\underline{x}_n \in \tilde{A}_{\delta_*}$ one derives the estimate

$$\begin{aligned} |x_{n+1}^{(\mu)} - x_{n+1}^{(\nu)}| &= \left| 2 - \frac{2\epsilon}{L} \sum_\rho \cos(2x_n^{(\rho)} - x_n^{(\nu)} - x_n^{(\mu)}) \frac{\sin(x_n^{(\mu)} - x_n^{(\nu)})}{x_n^{(\mu)} - x_n^{(\nu)}} \right| |x_n^{(\mu)} - x_n^{(\nu)}| \\ &= \left| 2 - \frac{2\epsilon}{L} \left[2\cos(x_n^{(\mu)} - x_n^{(\nu)}) + \sum_{\rho(\neq \nu,\mu)} \cos(2x_n^{(\rho)} - x_n^{(\mu)} - x_n^{(\nu)}) \right] \right. \\ &\quad \left. \cdot \frac{\sin(x_n^{(\mu)} - x_n^{(\nu)})}{x_n^{(\mu)} - x_n^{(\nu)}} \right| |x_n^{(\mu)} - x_n^{(\nu)}| \\ &\leq \left| 2 - 2\epsilon \left[\frac{2}{L} \cos(\delta_*) \frac{\sin(\delta_*)}{\delta_*} + \frac{L-2}{L} \cos(2\delta_*) \right] \right| |x_n^{(\mu)} - x_n^{(\nu)}| \quad . \end{aligned} \tag{17}$$

In the last step the inequalities $|2x^{(\rho)} - x^{(\mu)} - x^{(\nu)}| \leq 2\delta_* - |x^{(\mu)} - x^{(\nu)}|$ and $\cos(2\delta_* - x)\sin(x)/x \geq \cos(2\delta_*)$, $0 < x \leq \delta_*$ have been used. Choosing δ_* sufficiently small the prefactor becomes smaller than unity and the map contracts the neighbourhood $\tilde{A}_{\delta_*}$ for $\epsilon_c^s < \epsilon < \epsilon_c^m$ where δ_* obeys

$$2 - 2\epsilon \left[\frac{2}{L} \cos(\delta_*) \frac{\sin(\delta_*)}{\delta_*} + \frac{L-2}{L} \cos(2\delta_*) \right] < 1 \quad . \tag{18}$$

The same property holds for the corresponding neighbourhood A_{δ_*} of the diagonal in the phase space S^L. In terms of Fig.2 this set is contained in the interval whose final points are given by the two unstable fixed points. It is part of the basin of attraction of the synchronized state. From eq.(18) it is obvious that the diameter satisfies $\delta_* \sim (\epsilon - \epsilon_c^s)^{1/2}$ if the coupling approaches the critical value.

In principle one may proceed along the lines of section 2 to construct a symbolic dynamics on the complement set $A_{\delta_*}^C$. Instead of using the coordinate planes as the starting grid it seems to be more useful to start from a partition consisting of connected sets. Such a partition is induced for example by the planes $x^{(\nu)} = 2\pi k^{(\nu)}$, $1 \leq \nu \leq L-1$ and $\sum_{\nu=0}^{L-1} x^{(\nu)} = 2\pi k^{(0)}$ which contain the diagonal. The successive counterimages yield a partition of subsets of $A_{\delta_*}^C$ because $T(A_{\delta_*}) \subseteq A_{\delta_*}$ holds. In order to construct a unique symbolic dynamics one needs increasingly finer partitions. Unfortunately the uniform expansion of the map T on $A_{\delta_*}^C$ cannot be proven. However as long as $|\varphi^{(\nu)} - \varphi^{(\mu)}| \mathrm{mod} \pi > \delta_*$ holds for any pairs of indices then the estimate

$$\begin{aligned} \langle \underline{y} | D\tilde{T}(\underline{\varphi}) | \underline{y} \rangle &= 2\sum_\nu \left(y^{(\nu)}\right)^2 - \frac{\epsilon}{L} \sum_{\nu,\mu} \cos(2\varphi^{(\mu)} - 2\varphi^{(\nu)}) \left(y^{(\nu)} - y^{(\mu)}\right)^2 \\ &\geq 2\sum_\nu \left(y^{(\nu)}\right)^2 - 2\frac{\epsilon}{L}\cos(2\delta_*) \left[L\sum_\nu \left(y^{(\nu)}\right)^2 - \left(\sum_\nu y^{(\nu)}\right)^2 \right] \\ &\geq [2 - 2\epsilon\cos(2\delta_*)] \langle \underline{y} | \underline{y} \rangle \quad . \end{aligned} \tag{19}$$

shows that the map is expansive on a subset of $A_{\delta_*}^C$ because the inequalities (18) and $2 - 2\epsilon\cos(2\delta_*)$ can be satisfied simultaneously. This does not prove the existence of a unique chaotic repeller. But it supports the conjecture that at least in the vicinity of the instability $0 < \epsilon - \epsilon_c^s \ll 1$ the dynamics on $A_{\delta_*}^C$ is dominated by a chaotic repeller because the map is expansive on a large fraction of this set (cf. Fig. 2 for the case of two coupled maps where the existence of the repeller is evident).

Let me now study whether the conjecture made above fits in with numerical simulations for the time evolution of the coupled map lattice. Below the critical coupling strength ϵ_c^s the space–time–mixing state yields a quite trivial dynamics. The numerical simulations show a very noisy behaviour. Therefore I focus on the regime $\epsilon_c^s < \epsilon$. To visualize the time evolution of numerical simulations some global quantity is useful. Here the absolute value of the "mean field" $r = |\sum_\nu \exp(if(x^{(\nu)}))/L|$ (cf. section 4) has proven to be a suitable quantity. Its value fluctuates in the space time mixing regime and saturates to 1 in the synchronized state. A few typical time series are shown in Fig.3 for coupling strength $\epsilon = 0.65, 0.8$ and systems size $L = 15, 21$. $<$ Fig.3
They have been obtained from a randomly chosen initial condition. The time series splits into a random transient whose duration seems to increase with the system size and a sharp relaxation towards the synchronized solution. The sharpness of the transition enables us to introduce a well defined relaxation time

$$N_A(\underline{\varphi}) = \min\{n \in \mathbb{N}, | T^n(\underline{\varphi}) \in A\} \tag{20}$$

depending on the initial condition $\underline{\varphi}$ and some neighbourhood A of the synchronized state (e.g. the set (16)). Because of the fast relaxation this quantity is practically independent of the choice of A as far as this set is contained in the attracting region (16). The relaxation time fluctuates strongly as a function of the initial condition. Its distribution is given by

$$P_n(A) = \langle \delta(n, N_A(\underline{\varphi})) \rangle \tag{21}$$

where $\langle \ldots \rangle$ denotes an average over the distribution of initial points. The quantity (21) has been simulated numerically for various system lengths and coupling strength by taking an average over 10000 randomly chosen initial conditions. The relaxation time, that means the set A, has been defined by the condition that the mean field deviates from the final value 1 by less than 10^{-3}. Fig.4 shows some representative $<$ Fig.4 results. The distribution is in all cases Poisson like with an exponential tail. The least square fit between the maximum of the distribution and the first box that contains only one member of the ensemble is indicated also. The slope of this fit yields the tail of the distribution $P_n(A) \simeq C_1 \exp(-\lambda_{\epsilon,L} n)$ and coincides with the inverse of the mean relaxation time. The ϵ and L dependence of this quantity has been investigated from the numerical data. Fig.5 contains the dependence on the system size for various $<$ Fig.5 coupling strengths. The numerical capabilities restrict the analysis to small systems or coupling strength far above the critical value. However in the whole range a exponential dependence on the system size is clearly indicated, $\lambda_{\epsilon,L} \simeq \exp(-L\alpha_\epsilon)$. From this observation it is evident that large systems yield huge transient times. The slopes in Fig.5 allow for an analysis of the ϵ dependence also. The evaluation suggests a logarithmic dependence (cf. Fig.6) , $\alpha_\epsilon \simeq -\ln C_2 - a \ln(\epsilon - \epsilon_c^s)$, with $<$ Fig.6 a slope $a \sim 0.4$. Summarizing the numerical findings one observes a Poisson like distribution of relaxation times where the inverse of the mean relaxation time obeys $\lambda_{\epsilon,L} \simeq C_1[C_2(\epsilon - \epsilon_c^s)^a]^L$. Near the transition point ϵ_c^s and for large system size a huge space–time–mixing transient is observed.

These numerical findings are in accordance with the scenario conjectured above. First of all the Poisson like distribution of relaxation times is closely related to the decay rate of a chaotic repeller. In order to apply the conventional terminology of decay rates [19] it is necessary that the invariant set has vanishing Lebesgue measure. I do not have any proof of this property even for the simple model considered here. If we however assume that the dynamics for $\epsilon > \epsilon_c^s$ is governed by a repelling set then the shape of the distribution is an immediate consequence of a finite decay rate which coincides with the rate $\lambda_{\epsilon,L}$ (cf. appendix D). The decay rate can be estimated by a simple geometrical phase space argument. If one considers the motion on a space–time–mixing repeller as a stochastic process in phase space then the probability that a point falls on one step of iteration within the attracting region (16) of the synchronized solution is given by its volume relative to the whole phase space measure. It is simply estimated as $p \simeq 2\pi(\delta_*)^{L-1}/(2\pi)^L \sim (\delta_*/(2\pi))^L$. The probability that the dynamics settles on the synchronized state after n iteration steps is given by the Binomial distribution $P_n \simeq (1-p)^{n-1} p \sim \exp(-np)$ which reduces to a Poisson distribution for small probability p. Its decay rate p yields the exponential dependence on the system size. The coupling strength enters via the diameter δ_* of the attracting region. It has been mentioned already that it scales as $\delta_* \sim (\epsilon - \epsilon_c^s)^{1/2}$ in reasonable agreement with the numerical data.

The transition from the space–time–mixing to the synchronized state at ϵ_c^s is accompanied by long transients which are supposed to be caused by a space–time–mixing repeller with a small escape rate. The nature of this repeller makes it difficult to observe the transition in numerical simulations if the coupling is changed adiabatically. Furthermore for system size $L \gtrsim 30$ the transients may become longer than any reasonable observation time and no synchronization will be observed at all. In the thermodynamic limit $L \to \infty$ the space–time–mixing state becomes in a certain sense stable. The transition from the synchronized state to the space–time–mixing regime which is observed if one decreases the coupling strength is always a sharp transition because it is brought about by the local properties near the diagonal. Hence the synchronization is accompanied by a strong hysteresis effect.

4 Non–hyperbolic coupled maps

Let me now focus on the opposite case $a \lesssim 2$. At the beginning I should mention that to my best knowledge there are no general results available whether the single site map (2) is chaotic in this case. Nevertheless slight modifications of the approaches used e.g. in [20] and numerical simulations indicate that the map has no stable periodic orbits and possesses a smooth invariant density for $a < 2$ which develops a singularity $\sim |\varphi|^{-2/3}$ at $a = 2$[8]. In addition the results described below have also been found for the case of the "logistic map" $f(\varphi) = 2\varphi(2\pi - \varphi)/\pi$, $0 \leq \varphi \leq \pi$, $f(\varphi) = 2\pi - f(2\pi - \varphi)$, $\pi \leq \varphi \leq 2\pi$ which is obviously chaotic.

For non–hyperbolic coupled maps the expansivity is lost for very weak coupling. Hence even the weak coupling regime shows complex structures and it seems to be impossible to give a general description. But it has been stressed recently [13, 14, 15, 16] that there seems to exist some general feature among globally coupled non–hyperbolic maps on which I will focus in this section. It is related to an unusual system size dependence of the fluctuations of global quantities and has been termed "violation of the law of large numbers". It has been suggested that this effect is related to some hidden coherence in the coupled system. Partial explanations based on numerical evidence and stochastic arguments have been given. But there seems to exist no approach which explicitly refers to the non–hyperbolicity of the system. I will try to make some approach in this direction.

Let me first briefly review the phenomenological results of numerical simulations (see also [13]). Owing to the global coupling of the maps the interaction is caused via a mean field

$$r_n \exp\left(i\Phi_n\right) := \frac{1}{L}\sum_\nu \exp\left(if(\varphi_n^{(\nu)})\right) \tag{22}$$

[8] The subsequent considerations demand for a generating partition. Its existence is guaranteed e.g in the case of expanding maps, $|(f^{(n)})'| > 1$ for some $n \in \mathbb{N}$.

which is a global quantity. In terms of this field the dynamics of eq.(1) is written as

$$\varphi_{n+1}^{(\nu)} = F_n(f(\varphi_n^{(\nu)})), \quad F_n(\varphi) = \varphi + \epsilon r_n \sin(\Phi_n - \varphi) \tag{23}$$

where the interaction function F_n depends on the other lattice sites via eq.(22). As long as the motion is chaotic the map (23) resembles the structure of a stochastically forced single map. A naive view of the law of large numbers suggests that the fluctuations of the mean field, $\Delta_{cc} := \langle r^2 \cos^2(\Phi)\rangle - \langle r\cos(\Phi)\rangle^2$, $\Delta_{ss} := \langle r^2 \sin^2(\Phi)\rangle - \langle r\sin(\Phi)\rangle^2$, will decrease with the system size as $\sim L^{-1}$. This behaviour is found in the hyperbolic case $a \ll 2$. But in non–hyperbolic situations strong deviations are observed. Fig.7 shows a few representative results for coupling strength $\epsilon = 0.2$ and $<$ Fig.7 $a = 1.98$, $a = 2$. The fluctuations saturate at some finite value if the system size is increased beyond a critical value. The behaviour is more pronounced for the fluctuation Δ_{ss} and seems to increase by approaching $a = 2$ or increasing the coupling strength. Furthermore this "violation of the law of large numbers" seems to occur for infinitesimal coupling strength in the non–hyperbolic case $a = 2$. But it is difficult to obtain conclusive results from the numerical simulations because of limitations in the computational capabilities. A much more refined view is obtained if the whole distribution function of the mean field is considered. Fig.8 shows some typical re- $<$ Fig.8 sults which have been obtained from a time series of length $N = 2.5 \times 10^6$ and a random initial condition. For small system size the distribution resembles a Gaussian shape. If the critical system size is approached the peak broadens and a double peak structure develops if the size is increased further. The spacing of the two maxima is responsible for the saturation of the moment Δ_{ss}. It is furthermore instructive to look at the time evolution of the mean field (22). A finite part of the time series corresponding to parameter values used in Fig.8 is shown in Fig.9 . Whereas the $<$ Fig.9 evolution behaves random below the critical system size an intermittent behaviour is clearly observed in the critical region. Beyond the critical length a hopping between states of positive and negative phases Φ at a very low transition rate is detected. It may exceed the available time span of the numerical simulation. The numerical results resemble strongly the phenomenon of a symmetry breaking chaos transitions or crisis induced intermittency [21, 22, 23] which are well known in the context of low dimensional chaotic dynamics. An explanation in this direction will be developed in the sequel.

A partial analytical approach may be developed by studying the evolution of the full phase space distribution of the coordinates $\varphi^{(\nu)}$ which is governed by the Ruelle–Frobenius–Perron equation. This full description seems to be to complicated to handle with. But one may resort to a formulation introduced in ref.[24]. The global coupling enables us to reduce the full dynamics to the evaluation of a single site map (23) which depends on the mean field (22). Hence the motion can be analysed completely by considering the probability distribution that a value φ occurs at time n at a lattice site. It reads

$$\rho_n(\varphi) := \frac{1}{L}\sum_\nu \delta_{2\pi}(\varphi_n^{(\nu)} - \varphi) \qquad (24)$$

where $\delta_{2\pi}$ denotes the 2π periodic extension of the δ–distribution. An exact closed equation of motion for this density is easily written down. On one hand the density determines the mean field (22) via

$$r_n \exp(i\Phi_n) = \int \rho_n(\varphi)\exp(if(\varphi))\,d\varphi \quad . \qquad (25)$$

On the other hand eq.(23) yields for the time evolution

$$\begin{aligned} \rho_{n+1}(\varphi) &= \frac{1}{L}\sum_\nu \delta_{2\pi}[(F_n \circ f)(\varphi^{(\nu)}) - \varphi] \\ &= \int \delta_{2\pi}[(F_n \circ f)(\psi) - \varphi]\rho_n(\psi)\,d\psi \quad . \end{aligned} \qquad (26)$$

Eq.(26) resembles the structure of a Ruelle–Frobenius–Perron equation. But the dependence of the mean field, that means the map F_n, on the density itself turns the relation into a nonlinear evolution equation. On this level eq.(26) is an exact consequence of the full dynamical system (1) as long as the densities are of the form (24). For large system size these densities tend in a weak sense to smooth distributions for typical initial conditions. This suggestive observation has been checked numerically (cf. Fig.11). Mathematical rigorous proofs for hyperbolic systems may be developed along the lines of ref.[25]. Hence eq.(26) yields the appropriate description of the dynamics in the limit of large system size if one considers sufficiently smooth densities[9]. In contrast to the full Ruelle–Frobenius–Perron equation of the coupled map lattice. it allows for an analysis with reasonable effort.

The long time behaviour is of central interest. Therefore the discussion of stationary states and their stability will be discussed. To this end I briefly mention the symmetries of the coupled map lattice. The full system (1), (2), (3) is obviously invariant with respect to phase inversion, $\forall\nu, \varphi^{(\nu)} \to 2\pi - \varphi^{(\nu)}$. This property carries over to the mean field equation (26) which is invariant with respect to $\varphi \to 2\pi - \varphi$. Any solution is therefore either symmetric, $\rho_n^{(S)}(\varphi) = \rho_n^{(S)}(2\pi - \varphi)$ or has an inversion symmetric counterpart $\rho_n^{(-)}(\varphi) = \rho_n^{(+)}(2\pi - \varphi) \neq \rho_n^{(+)}(\varphi)$. In the former case the mean field is real, $\Phi_n^{(S)} = 0, \pi$, whereas in the latter case it takes complex conjugate values, $r_n^{(+)} = r_n^{(-)}$, $\Phi_n^{(+)} = -\Phi_n^{(-)}$. A real mean field does not destroy the inversion symmetry of the map (23) (cf. Fig.10) so that a symmetric solution stays symmetric <Fig.10
in the course of the dynamics.

[9]It should be remarked that eq.(26) does not depend on L explicitely. The system size enters only through the structure of the distribution. A smooth distribution corresponds to the limit $L \to \infty$

The stationary densities $\rho_*(\varphi)$ are determined by

$$\rho_*(\varphi) = \int \delta_{2\pi}[(F_* \circ f)(\psi) - \varphi]\rho_*(\psi)\, d\psi \tag{27}$$

where the map F_* depends on the field

$$r_* \exp(i\Phi_*) = \int \rho_*(\varphi) \exp(if(\varphi))\, d\varphi \quad . \tag{28}$$

Eq.(27) looks like a Ruelle–Frobenius–Perron equation for the one dimensional map $F_* \circ f$. The existence of smooth solutions and their computation is in general a difficult problem. At least for symmetric densities the results of [20] suggest that such a solution exists whereas numerical simulations indicate the existence of asymmetric solutions also for appropriate coupling strengths (cf. Fig.11). If one presupposes the existence of stationary smooth solutions their structure can be analysed for the non–hyperbolic case $a = 2$. In that case the point of slope zero, $\varphi = \pi$, introduces singularities into the density which can be computed by a perturbation expansion [26]. Symmetric densities are easy to analyse. Here the point of slope zero is mapped onto the unstable fixed point (cf. Fig.10), a situation which is identical to the Smale complete logistic equation. The density $\rho_*^{(S)}$ develops a singularity $\rho_*^{(S)}(\varphi) \sim |\varphi|^{-2/3}$ at $\varphi = 0$ and is otherwise smooth. The case of asymmetric solutions is a little more difficult to handle (cf. appendix E). Here the point of slope zero has a nontrivial trajectory, $\omega_n = (F_* \circ f)^n(\pi)$. The density develops singularities at these image points , $\rho_*^{(\pm)} \sim |\varphi - \omega_n|^{-2/3}$ whose strengths decrease exponentially with n[10]. Hence the structure may become quite complicated. The main feature of these results persist for a values close to the non–hyperbolic case, $2 - a \ll 1$. In that case the critical slope is nonzero but small and causes sharp peaks in the density in contrast to actual singularities. The symmetric solution is smooth and has a sharp maximum at the unstable fixed point whereas the asymmetric density develops peaks along the orbit of the critical point. For small coupling strength ϵ this orbit yields a sequence which departs exponentially from the unstable fixed point (cf. Fig.11 for comparison with numerical simulations). Finally it should be mentioned that the described enhancement of the density reflects a partial synchronization among lattice sites in the full system. This effect is attributed to the non–hyperbolicity of the single site map.

Some qualitative estimates on the stability of the stationary density will be developed in the sequel. As long as linear stability analysis is valid the eigenvalue problem which emerges from the linearization of the evolution equation (26) governs the stability. It reads

$$\lambda h_\lambda(\varphi) \quad = \quad \int \delta[\varphi - (F_* \circ f)(\psi)] h_\lambda(\psi)\, d\psi$$

[10] The result is rigorous if the orbit terminates at some unstable periodic orbit [20].

$$-\epsilon \mathrm{Im} \left(\int \delta'[\varphi - (F_* \circ f)(\psi)] \rho_*(\psi) \exp[-if(\psi)] \, d\psi \right.$$
$$\left. \cdot \int \exp[if(\psi)] h_\lambda(\psi) \, d\psi \right) \quad . \tag{29}$$

In order that this expression makes sense the density ρ_* should be differentiable. For that reason the subsequent discussion is restricted to the case of symmetric densities and $a < 2$. Integrating the eigenvalue equation (29) with respect to φ one obtains immediately that eigenvalues $\lambda \neq 1$ are necessarily related to eigenfunctions with vanishing average value, $\int h_\lambda(\varphi) \, d\varphi = 0$. Furthermore there exists an eigenvalue $\lambda = 1$ which is independent of the coupling strength ϵ. This "Goldstone mode" is related to a continuous symmetry of the fixed point equation (27)[11]. For small coupling eq.(29) yields a perturbation of the Ruelle–Frobenius–Perron equation of the symmetric map $F_* \circ f$. The unperturbed eigenvalue problem possesses by presupposition (cf. the remarks at the beginning of this section) one eigenvalue $\lambda = 1$. The remaining part of the spectrum has modulus smaller than 1 and is separated from the largest eigenvalue by a finite gap. This gap shrinks to zero if a approaches 2. The largest eigenvalue $\lambda = 1$ persists if one turns on a small coupling $\epsilon > 0$. A finite value is neccessary in order that additional eigenvalues cross the line $\mathrm{Re}(\lambda) = 1$ to induce the instability of the symmetric density. The strength of this critical coupling is supposed to tend to zero in the limit $a \uparrow 2$ because the gap vanishes in that limit. Hence in the non–hyperbolic case even an infinitesimal coupling ϵ may cause the instability of the symmetric density because the leading eigenvalue $\lambda = 1$ is degenerated with a continuous part of the spectrum [26, 27]. Summarizing this reasoning one may expect that the symmetric density, $\Phi_* = 0$, is stable up to a critical coupling strength. This critical value tends to zero in the limit $a \uparrow 2$.

Further analysis is possible if one looks at numerical solutions of the mean field equation (26). This analysis captures also dynamical properties of the system. To implement the numerical algorithm a sufficiently fine partition of the phase space $[0, 2\pi]$ in intervals of equal size has been considered. The densities ρ_n are approximated by step functions on this partition. Then eq.(26) turns into a matrix equation which can be iterated easily. For our purpose a number of $\sim 10^4$ boxes is sufficient to give accurate results for the densities too.

First of all it has been checked that the mean field equation yields the correct description in the limit of large system size. As a typical example Fig.11 displays <Fig.11 the stationary densities for two parameter combinations $\epsilon = 0.05$, $a = 1.99, 1.996$ in comparison with the direct numerical simulation of a huge coupled map lattice $L = 5 \times 10^5$. These densities have been obtained from the time evolution of the map lattice (1) and the mean field equation (26) which tend to a time independent state for the chosen parameter values after 100 time steps. The first set of parameters

[11] Denoting by $\rho_*(\varphi, \epsilon)$ a (non–normalized) solution of eq.(27) then $\alpha \rho_*(\varphi, \epsilon) = \rho_*(\varphi, \epsilon/\alpha)$, $\alpha > 0$ holds.

yields a symmetric density with a sharp maximum at the unstable fixed point whereas for the second combination of parameters a non–symmetric density is observed. Its maxima escape from the unstable fixed point with an exponential rate as predicted by the analytical analysis. The mean field result and the direct simulation coincide within the statistical errors which are caused by finite size effects. Hence the system shows the two types of stationary solutions described above. Beyond the stationary properties the mean field approach reproduces the dynamical features also. Fig.11c shows a typical time evolution of the phase of the mean field Φ_n obtained from a random initial condition, that means a constant initial density. Again the mean field description and the simulation coincide.

In a second step the stability of the symmetric solution in the weak coupling case was investigated. As analytical approaches are difficult to apply the results of a numerical integration of the mean field equation (26) are presented. For sufficiently small coupling ϵ one finds a stable symmetric stationary density for $a < 2$. If the coupling is increased this solution becomes unstable. Fig.12 shows the time evolution <Fig.12
of the phase of the mean field, Φ_n, that emerges from a slightly asymmetric initial condition. If we disregard a short oscillatory transient then an exponential relaxation towards the symmetric solution is observed below a critical coupling strength for $a < 2$. An exponential relaxation towards an asymmetric stationary state is observed if we cross the critical coupling strength. The critical value decreases if one approaches $a = 2$. In the non–hyperbolic case $a = 2$ for every coupling strength only a stable asymmetric state (cf. Fig.12d) was found. The whole scenario is consistent with the qualitative linear stability analysis presented above.

The numerical treatment of the mean field equation allows for an investigation of regions of larger coupling strength also which seem to be inaccessible by means of an analytical approach. The time evolution of the system becomes quite complicated. Typically I have observed a transition from stationary to oscillatory and time chaotic behaviour. As a typical example Fig.13 contains the time series of the phase of the <Fig.13
mean field for increasing coupling strength and two parameter values $a = 1.99, 1.95$. Although this scenario which resembles on a phenomenological level the transition to chaos in low dimensional dynamical systems [3] has been observed for almost all combinations of parameter values it is not clear to which extent it depends on the details of the system under consideration.

5 Conclusion

For a certain class of globally coupled map lattices analytical estimates together with numerical simulations has lead to a detailed insight into the dynamics. For coupled shift maps, $a = 0$ global synchronization sets in beyond the weak coupling regime which is governed by a space–time–mixing stationary. It is accompanied by transients whose length increase exponentially with the system size. The scenario extends to perturbations of the shift map, $a > 0$. But then there might occur a gap between the onset of synchronization at ϵ_c^s and the loss of space–time–mixing, that means the loss of local expansivity, at ϵ_c^e. It is however difficult to observe such phenomena in numerical simulations because of the tremendous transients. If the coupling is increased to very large values, $\epsilon \gg \epsilon_c^m$, then the synchronized state undergoes further bifurcations and may be destroyed again. But these effects seem to depend strongly on the nature of the coupling in contrast to the above mentioned properties.

A different behaviour is observed for strongly non–hyperbolic coupled maps, $a \lesssim 2$. For small coupling a symmetry breaking is observed in the thermodynamic limit of infinite system size. This bifurcation is triggered by the non–hyperbolic points and can be understood qualitatively in terms of a mean field transfer operator. Fluctuations which are introduced by finite size effects cause jumps between the different asymmetric states. They are responsible for the bimodal structure of the distribution of the mean field and the anomalous behaviour of the mean square deviations. If the size of the system is increased the time scale of the intermittent hopping may exceed the observation time so that the computed mean square deviation drops again. But then secondary bifurcations to time dependent stationary states lead to a lower bound for the mean square deviations. Whether the symmetry breaking of the inversion symmetry is a necessary condition for the subsequent bifurcations like in low dimensional systems [28] is an unsolved problem. Map lattices without inversion symmetry like the frequently discussed coupled logistic maps are believed to undergo these "secondary" bifurcations. This scenario will explain the so called "violation of the law of large numbers". Results will be published elsewhere.

Depending on the non–hyperbolicity of the single site map the breaking of the inversion symmetry precedes the appearance of a synchronized state. For intermediate values of the parameter a both bifurcations may interact. The consequences of such higher order codimension bifurcation is at the moment not clear. Furthermore it is an open question whether it influences the dynamics on a time scale that is observable in numerical simulations of large coupled map lattices.

The model system proposed in this publication constitutes an example which can be handled to some extent by analytical methods. It seems therefore promising to construct explicitly the two dimensional spin Hamiltonian and to analyse how the dynamical synchronization is related to phase transitions in the corresponding spin lattice. Such a relation might be helpful to understand pattern formation in more complicated and realistic systems. Work in this direction is in progress.

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Note added in proof

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Appendix A

For simplicity I will give the construction of a Markov partition for the case $L = 2$ only but using a notation which can be generalized to arbitrary L values of the system size.

A Markov partition consists of 2^L sets $U_{\underline{\sigma}}$, labeled by an index $\underline{\sigma} = \sigma_0, \sigma_1$, with the property that $T : U_{\underline{\sigma}} \to S^L$ is bijective. In addition boundaries have to be mapped on boundaries, $T(\partial U_{\underline{\sigma}}) \subseteq \bigcup \partial U_{\underline{\sigma}}$.

If one has in mind that the Markov partition of the uncoupled system is given by the coordinate planes one realizes that the Markov partition of the coupled system is generated from invariant codimension one manifolds in a neighbourhood of these planes. To be more definite one searches for manifolds $\mathcal{M}_0$ and $\mathcal{M}_1$

$$\mathcal{M}_0 := \{(h_0(t), h_1(t)) \,|\, h_0(t+2\pi) = h_0(t) + 2\pi, h_1(t+2\pi) = h_1(t)\} \tag{30}$$

$$\mathcal{M}_1 := \{(k_0(t), k_1(t)) \,|\, k_0(t+2\pi) = h_0(t), k_1(t+2\pi) = k_1(t) + 2\pi\} \tag{31}$$

which are invariant with respect to $\tilde{T}$. The 2π periodic extension of these manifolds leads to a coordinate grid. The counterimage of this grid with respect to $\tilde{T}$ induces by construction the Markov partition on S^L.

It is sufficient to construct $\mathcal{M}_0$ because by symmetry the second manifold follows via

$$k_0(t) = h_1(t), \quad k_1(t) = h_0(t) \quad . \tag{32}$$

To construct the manifold (30) the function space

$$\mathcal{L} = \{\underline{h} : \mathbb{R} \to \mathbb{R}^2 \,|\, h_0(t+2\pi) = h_0(t) + 2\pi, h_1(t+2\pi) = h_1(t), h_i \textit{ continuous}\} \tag{33}$$

is introduced. If $\|.\|$ denotes some vector norm this space becomes a Banach space with respect to the metric

$$d\left(\underline{h},\underline{\tilde{h}}\right) := \max_{t\in[0,2\pi]} \|\underline{h}(t) - \underline{\tilde{h}}(t)\| \quad . \tag{34}$$

The condition of invariance reads

$$h_\nu(s) = \left(\tilde{T}\left(\underline{h}(t)\right)\right)^{(\nu)} = \tilde{f}\left(h_\nu(t)\right) + \frac{\epsilon}{L}\sum_\mu g\left(\tilde{f}\left(h_\mu(t)\right) - \tilde{f}\left(h_\nu(t)\right)\right) \quad . \tag{35}$$

If one fixes the parametrization via the relation $s = 2t$ one is lead to consider the following operator

$$\left(\mathcal{K}\underline{h}\right)(t) := \tilde{T}^{-1}\left(\underline{h}(2t)\right) \quad . \tag{36}$$

It is defined for $0 \leq \epsilon < \epsilon_c^m$ because $\tilde{T}$ is invertible (cf. eq.(7)). The property $\tilde{f}(x+2\pi) = \tilde{f}(x) + 4\pi$ implies that $\mathcal{K}$ maps $\mathcal{L}$ into $\mathcal{L}$.

A fixed point of $\mathcal{K}$ determines the manifold $\mathcal{M}_0$

$$\underline{h}(t) = \left(\mathcal{K}\underline{h}\right)(t) = \tilde{T}^{-1}\left(\underline{h}(2t)\right) \quad . \tag{37}$$

For $0 \leq \epsilon < \epsilon_c^e$ the map $\tilde{T}$ is expansive (cf. eq.(14)). Hence the inverse is a contraction, a property which carries over to the operator $\mathcal{K}$ owing to the definition (36)

$$d\left(\mathcal{K}\underline{h},\mathcal{K}\underline{\tilde{h}}\right) \leq c^{-1} d\left(\underline{h},\underline{\tilde{h}}\right), \quad c > 1 \quad . \tag{38}$$

By the contraction principle the fixed point equation (37) admits a unique solution which can be obtained by iteration

$$\underline{h}^{(n+1)} = \mathcal{K}\underline{h}^{(n)}, \quad \lim_{n\to\infty} \underline{h}^{(n)} = \underline{h} \quad . \tag{39}$$

It remains to show that the manifolds (30) and (31) intersect on a codimension two manifold, e.g. at one point. First of all they intersect because both manifolds extend to infinity along different coordinate axis. On the other hand the intersection has to obey $t = 0$. Suppose that $t_* \neq 0$ yields a point of intersection $\underline{h}(t_*) = \underline{k}(t_*)$. The invariance of both manifolds and eq.(37) imply that any point $2^k t_*$, $k \in \mathbb{Z}$ yields an intersection point, e.g.

$$h_0(2^k t_*) = k_0(2^k t_*) = h_1(2^k t_*) \tag{40}$$

But the left hand side tends to infinity for increasing k whereas the right hand side is bounded (cf. eq.(30)).

The degree of "smoothness" of the manifold (30) seems to decrease if one approaches the critical point ϵ_c^e. Fig.14 shows a few numerical solutions of eq.(37) for <
the choice of coupled shift maps, $\tilde{f}(x) = 2x$, $g(x) = \sin(x)$. In fact one can show Fig.14

after some algebra that the manifold obeys $h_{0/1}(t) = (t \mp h(t))/2$ where h is Hölder continuous with exponent α, $2^\alpha < 2 - 2\epsilon$.

In view of eq.(39) the boundary of the Markov partition can be generated as a successive counterimage of the coordinate planes. The partitions $U^{(n)}_{\underline{\sigma}_0,\underline{\sigma}_1,\ldots}$ used in section 2 for the construction of the symbolic dynamics are generated in the same fashion. Hence both approaches are strongly correlated to each other.

Appendix B

The determinant of the expression (5) reads

$$\det(D\tilde{T}(\underline{x}) = \prod_{\nu=0}^{L-1} f'(x^{(\nu)}) \det(\underline{\underline{A}}) \tag{41}$$

where the matrix on the right hand side is given by

$$A_{\nu\mu} = \delta_{\nu\mu} - \frac{\epsilon}{L} \sum_\rho \gamma_{\nu\rho} \delta_{\nu\mu} + \frac{\epsilon}{L} \gamma_{\nu\mu} \quad . \tag{42}$$

For simplicity the abbreviation

$$\gamma_{\nu\mu} := g' \left(f(x^{(\mu)}) - f(x^{(\nu)}) \right) = \cos \left((f(x^{(\mu)}) - f(x^{(\nu)}) \right) \tag{43}$$

has been used. We calculate the matrix of the expression (42) via its eigenvalue equation. It is written as

$$\Lambda_\nu v^{(\nu)} = \frac{\epsilon}{L} \left(c_\nu \sum_\mu c_\mu v^{(\mu)} + s_\nu \sum_\mu s_\mu v^{(\mu)} \right) \tag{44}$$

where use has been made of the abbreviations

$$\begin{aligned} \Lambda_\nu &:= \lambda - 1 + \frac{\epsilon}{L} \sum_\mu \gamma_{\nu\mu} \\ c_\nu + i s^{(\nu)} &:= \exp(i f(x^{(\nu)})) \end{aligned} \tag{45}$$

and the trigonometric identity

$$\gamma_{\nu\mu} = c_\nu c_\mu + s_\nu s_\mu \quad . \tag{46}$$

λ denotes the eigenvalue and $v^{(\nu)}$ the components of the eigenvector. Introducing the two real quantities α, β

$$\alpha + i\beta := \sum_\mu (c_\mu + i s_\mu) v^{(\mu)} \tag{47}$$

the eigenvalue equation (44) reduces to

$$\begin{aligned} \alpha &= \frac{\epsilon}{L}\sum_\nu c_\nu \frac{c_\nu\alpha + s_\nu\beta}{\Lambda_\nu} \\ \beta &= \frac{\epsilon}{L}\sum_\nu s_\nu \frac{c_\nu\alpha + s_\nu\beta}{\Lambda_\nu} \quad . \end{aligned} \tag{48}$$

The condition for a nontrivial solution of this homogeneous linear equation yields a polynomial of degree L in the eigenvalue λ

$$\prod_\nu \Lambda_\nu - \frac{\epsilon}{L}\sum_\nu \prod_{\rho(\neq\nu)} \Lambda_\rho + \frac{1}{2}\left(\frac{\epsilon}{L}\right)^2 \sum_{\nu\neq\rho}(1-\gamma_{\nu\rho}^2) \prod_{\mu(\neq\nu,\rho)} \Lambda_\mu = 0 \quad . \tag{49}$$

Eq.(49) is the characteristic polynomial of the matrix (42). Hence its value at $\lambda = 0$ yields the determinant of the matrix. Combining this expression with eq.(41) leads to the result (6).

Appendix C

The abbreviation $A \overset{T}{\leftrightarrow} B$ is used throughout this paragraph to indicate that the map $T : A \to B$ is bijective.

Let me start with the partition $\{U^{(0)}_{\underline{\sigma}_0}\}$ which obeys $U^{(0)}_{\underline{\sigma}_0} \overset{T}{\leftrightarrow} S^L$. Consider a fixed phase space point $\underline{\varphi}$. It possesses 2^L counterimages, one of them in every set of the partition. They are labeled according to the rule $\underline{\varphi}_{\underline{\sigma}_0} \in U^{(0)}_{\underline{\sigma}_0}$. Consider for fixed index $\underline{\sigma}_1$ the counterimages of $\underline{\varphi}_{\underline{\sigma}_1}$. Again every set of the partition contains exactly one of these counterimages and indices are assigned according to $\underline{\varphi}_{\underline{\sigma}_0,\underline{\sigma}_1} \in U^{(0)}_{\underline{\sigma}_0}$.

The counterimage of $U^{(0)}_{\underline{\sigma}_1}$ is given by $\bigcup_{\underline{\sigma}} U^{(1)}_{\underline{\sigma},\underline{\sigma}_1}$ and each connected component obeys $U^{(1)}_{\underline{\sigma},\underline{\sigma}_1} \overset{T}{\leftrightarrow} U^{(0)}_{\underline{\sigma}_0}$. Hence it contains exactly one of the second counterimages $\underline{\varphi}_{\underline{\sigma}_0,\underline{\sigma}_1}$ and the index $\underline{\sigma} = \underline{\sigma}_0$ is assigned to this component. Thus the relations

$$U^{(n)}_{\underline{\sigma}_0,\ldots,\underline{\sigma}_n} \overset{T}{\leftrightarrow} U^{(n-1)}_{\underline{\sigma}_1,\ldots,\underline{\sigma}_n} \tag{50}$$

$$V^{(n-1)}_{\underline{\sigma}_0,\ldots,\underline{\sigma}_{n-1}} := \bigcup_{\underline{\sigma}_n} U^{(n)}_{\underline{\sigma}_0,\ldots,\underline{\sigma}_n} \tag{51}$$

$$U^{(n-1)}_{\underline{\sigma}_0,\ldots,\underline{\sigma}_{n-1}} \cap V^{(n-1)}_{\underline{\sigma}_0,\ldots,\underline{\sigma}_{n-1}} \neq \emptyset \tag{52}$$

are proven for $n = 1$.

One proceeds by induction. Take one point $\underline{\psi}$ from the set (52) and denote its 2^L counterimages by $\underline{\psi}_{\underline{\sigma}}$. The counterimage of $U^{(n-1)}_{\underline{\sigma}_1,\ldots,\underline{\sigma}_n}$ is given by $\bigcup_{\underline{\sigma}_0} U^{(n)}_{\underline{\sigma}_0,\ldots,\underline{\sigma}_n}$ and

$U^{(n)}_{\underline{\sigma}_0,\ldots,\underline{\sigma}_n} \overset{T}{\leftrightarrow} U^{(n-1)}_{\underline{\sigma}_0,\ldots,\underline{\sigma}_{n-1}}$. Therefore each component contains exactly one of the counterimages which we denote by $\underline{\psi}_{\underline{\sigma}_0}$. The counterimage of the other set $V^{(n-1)}_{\underline{\sigma}_1,\ldots,\underline{\sigma}_n}$ is by construction given by $\bigcup_{\underline{\sigma}} V^{(n)}_{\underline{\sigma},\underline{\sigma}_1,\ldots,\underline{\sigma}_n}$ and $V^{(n)}_{\underline{\sigma},\underline{\sigma}_1,\ldots,\underline{\sigma}_n} \overset{T}{\leftrightarrow} V^{(n-1)}_{\underline{\sigma}_1,\ldots,\underline{\sigma}_n}$. Hence every component of the counterimage contains exactly one of the counterimage points and one fixes the index $\underline{\sigma}$ according to the rule $\underline{\psi}_{\underline{\sigma}_0} \in V^{(n)}_{\underline{\sigma}_0,\ldots,\underline{\sigma}_n}$. But then relations (50–52) are obtained where n is replaced by $n+1$.

Appendix D

Let A denote a neighbourhood of the attractor. Then eqs.(20) and (21) immediately lead to

$$\sum_{k=0}^{n} P_k(A) = \lambda(T^{-n}(A)) \tag{53}$$

where λ denotes the normalized measure which yields the average in eq.(21). A natural choice for this measure is the normalized Lebesgue measure which corresponds to a uniform distribution of initial conditions. Let us presuppose that the quantity (53) tends to 1 in the limit $n \to \infty$. It means that the system has only one attracting set and that all repellers have (Lebesgue) measure zero. From the normalization of the measure and the identity $T^{-n}(S^L/A) = S^L/T^{-n}(A)$ one obtains

$$\sum_{k=n+1}^{\infty} P_k(A) = \lambda(T^{-n}(S^L/A)) \quad . \tag{54}$$

The conventional definition of the escape rate from a repelling set Λ is given by [19]

$$\sigma = -\lim_{U_\epsilon \downarrow \Lambda} \lim_{n\to\infty} \frac{1}{n} \ln \lambda(T^{-n}(U_\epsilon) \tag{55}$$

where U_ϵ denotes an ϵ neighbourhood of the repelling set. It is assumed that the escape rate does not depend on the choice of the (sufficiently small) neighbourhood so that one limit can be suppressed. The set S^L/A constitutes a suitable neighbourhood. Then eq.(55) yields the asymptotic behaviour

$$\lambda(T^{-n}(S^L/A)) \simeq C_1 \exp(-n\sigma) \quad . \tag{56}$$

Eqs.(54) and (56) result in the asymptotic relation

$$P_n(A) \simeq C_2 \exp(-n\sigma) \quad . \tag{57}$$

Appendix E

Eq.(27) can be viewed as a Ruelle–Frobenius–Perron equation for the asymmetric map $F_* \circ f$. It may be solved by iterating a smooth initial density $h_{n=0}(\varphi)$ until a stationary state is reached [26]

$$h_{n+1}(\varphi) = \int \delta_{2\pi}[(F_{\Phi_*} \circ f)(\psi) - \varphi] h_n(\psi)\, d\psi \quad . \tag{58}$$

For later use the explicit dependence of the map F_* on the phase of the mean field, Φ_*, has been indicated. Evaluating the integral the relation can be written as

$$h_{n+1}(\varphi) = \frac{h_n[(F_{\Phi_*} \circ f)^{-1}(\varphi)]}{(F_{\Phi_*} \circ f)'[(F_{\Phi_*} \circ f)^{-1}(\varphi)]} + \frac{h_n[2\pi - (F_{-\Phi_*} \circ f)^{-1}(2\pi - \varphi)]}{(F_{-\Phi_*} \circ f)'[(F_{-\Phi_*} \circ f)^{-1}(2\pi - \varphi)]} \tag{59}$$

where the the function f is restricted to the interval $[0, \pi]$. If the initial function $h_{n=0}(\varphi)$, is smooth the first iterate develops singularities at the critical points of the denominator. They are determined by

$$\begin{aligned} (F_{\Phi_*} \circ f)^{-1}(\varphi) \uparrow \pi \quad &\Leftrightarrow \quad \varphi \uparrow \omega_1 \\ (F_{-\Phi_*} \circ f)^{-1}(2\pi - \varphi) \uparrow \pi \quad &\Leftrightarrow \quad x \downarrow \omega_1 \quad . \end{aligned} \tag{60}$$

where $\omega_1 := \epsilon r_* \sin(\Phi_*)$ denotes the first image of the critical point. The strength of the singularity can be evaluated easily by considering a neighbourhood of this point.

i) $\varphi \downarrow \omega_1$: Straightforward Taylor series expansion yields to the leading order

$$\begin{aligned} (F_{\Phi_*} \circ f)^{-1}(\varphi) &= \frac{\varphi - \omega_1}{4F'_{\Phi_*}(0)} + O(2) \\ (F_{\Phi_*} \circ f)'[(F_{\Phi_*} \circ f)^{-1}(\varphi)] &= 4F'_{\Phi_*}(0) + O(1) \\ (F_{-\Phi_*} \circ f)^{-1}(2\pi - \varphi) &= \pi - \left(3\frac{\varphi - \omega_1}{F'_{-\Phi_*}(0)}\right)^{1/3} + O(2/3) \\ (F_{-\Phi_*} \circ f)'[(F_{-\Phi_*} \circ f)^{-1}(2\pi - \varphi)] &= F'_{-\Phi_*}(0)\left(3\frac{\varphi - \omega_1}{F'_{-\Phi_*}(0)}\right)^{2/3} + O(1) \end{aligned} \tag{61}$$

where $O(n)$ denotes contributions of order $(\varphi - \omega_1)^n$. Then eq.(59) reads

$$h_{n+1}(\varphi) \simeq \frac{h_n[(\varphi - \omega_1)/(4F'_{\Phi_*}(0))]}{4F'_{\Phi_*}(0)} + \frac{h_n(\pi)}{F'_{-\Phi_*}(0)[3(\varphi - \omega_1)/F'_{-\Phi_*}(0)]^{2/3}} \quad . \tag{62}$$

ii) $\varphi \uparrow \omega_1$: An analogous expansion leads to

$$(F_{\Phi_*} \circ f)^{-1}(\varphi) = \pi - \left(3\frac{\omega_1 - \varphi}{F'_{\Phi_*}(0)}\right)^{1/3} + O(2/3)$$

$$
\begin{aligned}
(F_{\Phi_*}\circ f)'[(F_{\Phi_*}\circ f)^{-1}(\varphi)] &= F'_{\Phi_*}(0)\left(3\frac{\omega_1-\varphi}{F'_{\Phi_*}(0)}\right)^{2/3}+O(1)\\
(F_{-\Phi_*}\circ f)^{-1}(2\pi-\varphi) &= \frac{\omega_1-\varphi}{4F'_{-\Phi_*}(0)}+O(2)\\
(F_{-\Phi_*}\circ f)'[(F_{-\Phi_*}\circ f)^{-1}(2\pi-\varphi)] &= 4F'_{-\Phi_*}(0)+O(1)
\end{aligned}
\tag{63}
$$

and

$$
h_{n+1}(\varphi)\simeq\frac{h_n(\pi)}{F'_{\Phi_*}(0)[3(\omega_1-\varphi)/F'_{\Phi_*}(0)]^{2/3}}+\frac{h_n[2\pi-(\omega_1-\varphi)/(4F'_{-\Phi_*}(0))]}{4F'_{-\Phi_*}(0)}\quad . \tag{64}
$$

Eqs.(62) and (64) clearly show that the first iterate develops a singularity $|\varphi-\omega_1|^{-2/3}$. On further iteration this singularity is carried along the trajectory of ω_1. With $\omega_2=(F_*\circ f)(\omega_1)$ eq.(58) yields in the vicinity of the image point

$$
\begin{aligned}
h_{n+1}(\varphi) &\sim \int\delta_{2\pi}[(F_*\circ f)(\psi)-\varphi]|\psi-\omega_1|^{-2/3}\,d\psi\\
&\sim |(F_*\circ f)'(\omega_1)|^{-1/3}|\varphi-\omega_2|^{-2/3}\quad .
\end{aligned}
\tag{65}
$$

Hence the strength of the singularity decreases by a factor $|(F_*\circ f)'(\omega_1)|^{-1/3}\sim 4^{-1/3}$ along the trajectory ω_n.

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Figure captions

Fig.1 Diagrammatic view of the partition $U^{(0)}_{\sigma^{(0}\sigma^{(1)}}$ for two coupled maps ($L = 2$) in the extended phase space. The dotted lined indicates the partition for the uncoupled case $\epsilon = 0$, and the broken line the partition for finite coupling $\epsilon > 0$.

Fig.2 Diagrammatic view of the map $f^{(-)}(x^{(-)}) = 2x^{(-)} - \epsilon \sin(2x^{(-)}) \; |\mathrm{mod}\, 4\pi$ for $\epsilon > \epsilon_c^s$. The box indicates the region which contains the repelling Cantor set. Regions which are locally contracting escape from the box in one iteration step.

Fig.3 Time series of the amplitude of the mean field for different coupling strengths and system sizes. (a) $\epsilon = 0.65$, $L = 15$, (b) $\epsilon = 0.65$, $L = 21$, (c) $\epsilon = 0.8$, $L = 15$, and (d) $\epsilon = 0.8$, $L = 21$.

Fig.4 Distribution of the relaxation times for $\epsilon = 0.8$ and (a) $L = 12$, (b) $L = 15$. The broken line indicates the least square fit. It is shifted by a factor two for clarity.

Fig.5 Dependence of the inverse of the mean relaxation time on the system length for several coupling strengths.

Fig.6 Dependence of the slopes in Fig.5 on the coupling strength $\epsilon - \epsilon_c^s$. The full line indicates the least square fit.

Fig.7 Mean square deviations Δ_{cc} (———) and Δ_{ss} (– – – –) in dependence on the system size for two values of the nonlinearity parameter $a = 1.98$ ($\times$) and $a = 2$ ($\Box$).

Fig.8 Distribution function of the mean field for $a = 1.98$, $\epsilon = 0.2$ and several values of the system size, (a) $L = 320$, (b) $L = 640$, and (c) $L = 1280$.

Fig.9 Time evolution of the mean field corresponding to the parameter values chosen in Fig.8, $a = 1.98$, $\epsilon = 0.2$ and (a) $L = 320$, (b) $L = 640$, (c) $L = 1280$. The upper chain of dots indicates the real part $r_n \cos(\Phi_n)$ and the lower chain the imaginary part $r_n \sin(\Phi_n)$.

Fig.10 Diagrammatic view of the stationary mean field map $F_* \circ f$ for (a) $\Phi_* = 0$ and (b) $\Phi_* > 0$. The broken line indicates the orbit of the critical point.

Fig.11 (a) Symmetric distribution of the phase space coordinates for $a = 1.99$ and $\epsilon = 0.05$. The full line indicates the result of a simulation of a map lattice of size $L = 5 \times 10^5$ whereas the broken line shows the mean field distribution. The latter curve is shifted by a factor 1/3 for clarity because both curves would coincide otherwise. 10^3 boxes (simulation) respectively 10^4 boxes (mean field) are used to figure the distribution.

(b) Asymmetric distribution for $a = 1.996$, $\epsilon = 0.05$ on a double logarithmic scale. The lower broken line indicates the mean field result whereas the upper broken line is shifted by a factor 3. 10^4 boxes (simulation) respectively 4×10^4 boxes (mean field) are used to figure the distribution.

(c) Time dependence of the phase of the mean field for parameter values used in Fig.11b, $a = 1.996$, $\epsilon = 0.05$. The full line indicates the mean field result whereas the broken line the simulation of the map lattice ($L = 5 \times 10^5$).

Fig.12 Time evolution of the phase of the mean field for several values of the coupling strength and nonlinearity parameter, (a) $a = 1.97$, (b) $a = 1.98$, (c) $a = 1.99$, and (d) $a = 2$.

Fig.13 Time evolution of the phase of the mean field for larger coupling strength.

(a) $a = 1.99$. The coupling strength is chosen as (from bottom to top) $\epsilon = 0.1$, 0.125, 0.15, 0.175, 0.2. The curves are shifted against each other by an offset of 0.1.

(b) $a = 1.95$. The coupling strength is chosen as (from bottom to top) $\epsilon = 0.4$, 0.5, 0.6, 0.7, 0.8. The curves are shifted against each other by an offset of 0.4.

Fig.14 Boundary of an element of a Markov partition for two coupled shift maps and several values of the coupling strength.