General Derivation of Scaling Functions near q–Phase Transition Points

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January 21, 1998

Abstract

Scaling relations of characteristic functions in the context of the thermodynamical formalism of dynamical systems are considered. Starting from the characteristic equation of the transfer operator a Ginzburg–Landau like expansion in the vicinity of a bifurcation point is proposed. The phase transition based on a doubly degenerated eigenvalue is discussed in detail and explicit expressions for the scaling functions are derived.

*A part of this work was performed within a program of the Sonderforschungsbereich 185 Darmstadt–Frankfurt at the TH Darmstadt (Germany).

1 Introduction

The thermodynamical formalism of dynamical systems constitutes a powerful tool for describing the complicated behaviour of nonlinear dynamical systems [1, 2]. It is mainly based on the analysis of the fluctuations of some variable $u(x)$ which arise via the time evolution in the equation under consideration [2]. The main properties of these fluctuations are contained in the eigenvalue equation of a certain kind of transfer operator [3]

$$\mathcal{H}_q^u h_q^{(l)} = \nu_q^{(l)} h_q^{(l)} \tag{1}$$

that depends on a real perameter q, which is often identified with the temperature in the context of the thermodynamical formalism of dynamical systems. In the special case of a discrete not necessarily one dimensional dynamical system $x_{n+1} = T(x_n)$ the explicit expression of the transfer operator reads [3, 4]

$$(\mathcal{H}_q^u h)(x) := \int \delta(x - T(y)) \exp(qu(y)) h(y) dy \quad . \tag{2}$$

It acts on the space of integrable functions. For simplicity we have assumed a discrete spectrum in eq.(1) which should be ordered according to the relation $\nu_q^{(0)} > |\nu_q^{(l)}| \geq |\nu_q^{(l')}|$, $0 < l \leq l'$. The importance of the transfer operator (2) and the eigenvalue equation (1) is emphasized by the fact that they allow for the expansion of the following characteristic function [3]

$$\langle \exp\left(q \sum_{i=0}^{n-1} u(T^i(x)) \right) \rangle = \left(\nu_q^{(0)} \right)^n \left\{ J_q^{(0)} + \sum_{l=1}^{\infty} J_q^{(l)} \left(\frac{\nu_q^{(l)}}{\nu_q^{(0)}} \right)^n \right\} \quad . \tag{3}$$

In this equation $J_q^{(l)}$ denote some expansion coefficients and the average $\langle \ldots \rangle$ is meant with respect to a distribution of initial points which is usually assumed to be the natural one (SRB measure). From eq.(3) one gets the well known relation for the characteristic function (topological pressure) [3, 5]

$$\phi(q) := \lim_{n\to\infty} \frac{1}{n} \langle \exp\left(q \sum_{i=0}^{n-1} u(T^i(x)) \right) \rangle = \ln \nu_q^{(0)} \tag{4}$$

which determines the stationary fluctuations of $u(x)$ as soon as the damping rates $\gamma_q^{(l)}$ and the frequencies $\omega_q^{(l)}$ which characterize the temporal correlations [3, 6]

$$\gamma_q^{(l)} + i\omega_q^{(l)} = -\ln \frac{\nu_q^{(l)}}{\nu_q^{(0)}} \quad . \tag{5}$$

In this sense the transfer operator (2) contains the long–time as well as the correlation characteristics of the nonlinear dynamical system.

It is well known that at a bifurcation point of a dynamical system a phase transition, that means a nonanalyticity in q in the quantities (4) and (5), might occur which is based on a degeneracy of the largest eigenvalues of the transfer operator (cf. Fig.1) [7]. If one approaches the bifurcation point from one side the degeneracy is removed and a typical scaling behaviour in the quantities of interest emerges [8, 9]. It is the main aim of this letter to analyse this behaviour from a rather general point of view. We will start our discussion from the characteristic equation of the transfer operator (2). It possesses the general form [1]

$$P(\nu_q, \delta, q) = 0 \quad . \tag{6}$$

Here $\delta \geq 0$ denotes a bifurcation parameter leading to a bifurcation at $\delta \downarrow 0$. If the system under consideration admits an attracting set then it is well known (cf. eq.(4)) that at $q = 0$ the largest eigenvalue equals unity for arbitrary values of the bifurcation parameter δ [10]. This observation results in the general property

$$P(1, \delta, q = 0) = 0 \tag{7}$$

which will be of some importance in the subsequent discussion. For simplicity we will restrict our analysis to the case that at the bifurcation point two simple eigenvalues become degenerated $\nu_{q_0}^{(0)}(\delta \downarrow 0) = \nu_{q_0}^{(1)}(\delta \downarrow 0)$ leading to a phase transition at $q = q_0$. They should be well separated from the remaining part of the spectrum (Fig.1). Refering to this situation and to analyticity assumptions in eq.(6) we are able to derive the correct scaling behaviour in the vicinity of the bifurcation point. In this sense our approach is similar to the Ginzburg–Landau treatment of second order phase transition.

In Section 2 we introduce the main ideas on the expansion of eq.(6). The algebraic calculations for two essential different cases arising in a natural way from our analysis are performed in the third and fourth section leading to the scaling behaviour mentioned above. Section 5 contains the discussion of two simple illustrative examples. Finally the results will be summarized.

2 Expansion of the characteristic equation

Let us introduce the abbreviation

$$\nu_c := \nu_{q_0}(\delta \downarrow 0) \tag{8}$$

for the eigenvalue at the phase transition point. Eq.(6) yields by presupposition two real eigenvalues which are well separated from the remaining part of the spectrum. Therefore a polynomial of second order can be extracted from P and eq.(6) reads

$$0 = P(\nu_q, \delta, q) = \left\{ \left(\frac{\nu_q}{\nu_c} - 1 \right)^2 + f(\delta, q) \left(\frac{\nu_q}{\nu_c} - 1 \right) + g(\delta, q) \right\} \bar{P}(\nu_q, \delta, q) \quad . \tag{9}$$

Here $\bar{P}$ does not vanish in the vicinity of the transition point $\nu_q = \nu_c$, $q = q_0$, $\delta = 0$. To simplify the following considerations the first part $\{\ldots\}$ has been written as a polynomial

[1] ν_q depends of course on δ, $\nu_q(\delta)$. In the most cases this dependence will be suppressed in the notation.

in $\nu_q/\nu_c - 1$ instead of ν_q. As the largest eigenvalue $\nu_q^{(0)}$ is positive the two largest solutions of eq.(9) are real. This implies that in the present case the frequency $\omega_q^{(1)}$ corresponding to the smallest damping rate vanishes (cf. eq(5)). Now the presupposition that at $q = q_0$, $\delta \downarrow 0$ eq.(9) yields the doubly degenerated eigenvalue ν_c sets in. It results in the relations

$$f(\delta \downarrow 0, q_0) = 0, \qquad g(\delta \downarrow 0, q_0) = 0 \quad . \tag{10}$$

Up to this moment no approximation was introduced into our considerations. We are interested in a vicinity of the phase transition point. Refering back to the third presupposition, the analyticity of the characteristic equation in δ and q, we are able to write down the following expansions of the prefactors f and g

$$\begin{aligned} f(\delta, q) &= f_{10}\delta + f_{01}(q - q_0) + f_{20}\delta^2 + f_{11}\delta(q - q_0) + f_{02}(q - q_0)^2 + O_3 \\ g(\delta, q) &= g_{10}\delta + g_{01}(q - q_0) + g_{20}\delta^2 + g_{11}\delta(q - q_0) + g_{02}(q - q_0)^2 + O_3 \quad . \end{aligned} \tag{11}$$

Here f_{ij} and g_{ij} denote the expansion coefficients and $O_3 := O(\delta^3, \delta^2(q - q_0), \delta(q - q_0)^2, (q - q_0)^3)$ the contributions of third and higher order. The values of g_{10} and g_{01} are restricted by the presupposition that eq.(9) yields real solutions. This can be shown easily by considering the following special situations:

i) $\delta \downarrow 0$: Then eq.(11) reads

$$\begin{aligned} f(\delta \downarrow 0, q) &= O(q - q_0) \\ g(\delta \downarrow 0, q) &= g_{01}(q - q_0) + O((q - q_0)^2) \end{aligned} \tag{12}$$

and eq.(9) yields

$$\left(\frac{\nu_q}{\nu_c} - 1\right)^2 + g_{01}(q - q_0) + O((q - q_0)^2, (q - q_0)\left(\frac{\nu_q}{\nu_c} - 1\right)) = 0 \quad . \tag{13}$$

As the the eigenvalues are real the inequality $(\nu_q/\nu_c - 1)^2 \geq 0$ holds. But as $q - q_0$ can take positive and negative values the relation

$$g_{01} = 0 \tag{14}$$

follows.

ii) $q = q_0$: Now eq.(11) reads

$$\begin{aligned} f(\delta, q_0) &= O(\delta) \\ g(\delta, q_0) &= g_{10}\delta + O(\delta^2) \end{aligned} \tag{15}$$

and eq.(9) yields

$$\left(\frac{\nu_q}{\nu_c} - 1\right)^2 + g_{10}\delta + O(\delta^2, \delta\left(\frac{\nu_q}{\nu_c} - 1\right)) = 0 \quad . \tag{16}$$

The same reasoning as above leads to

$$g_{10} \leq 0 \quad . \tag{17}$$

The difference between eqs.(14) and (17) follows from the fact that the bifurcation parameter δ can take only non negative values.

iii) If the phase transition emerges at $q_0 = 0$ and the system admits an attracting set then the corresponding critical eigenvalue is given by $\nu_c = 1$. Inserting this into eq.(9) and evaluating it at $q = 0$ with the help of eq.(7) one gets

$$g(\delta, q = q_0 = 0) = 0 \quad . \tag{18}$$

Inspecting eq.(11_2) the expansion coefficients read

$$g_{10} = 0, \qquad g_{20} = 0, \qquad (q_0 = 0) \quad . \tag{19}$$

With these settings only algebraic manipulations are needed to obtain the desired scaling behaviour from eqs.(9),(11), (14),(17) and (19). In doing this the two different cases $g_{10} = 0$ and $g_{10} < 0$ which correspond to $q_0 = 0$ and $q_0 \neq 0$ respectively have to be considered separately.

3 Case A: $q_0 = 0$

Then eq.(11) reads taking eqs.(14) and (19) into account

$$\begin{aligned} f(\delta, q) &= f_{10}\delta + f_{01}q + O(\delta^2, \delta q, q^2) \\ g(\delta, q) &= g_{11}\delta q + g_{02}q^2 + O_3 \quad . \end{aligned} \tag{20}$$

Introducing the scaling abbreviations

$$\nu_q - 1 =: \delta\, \psi_\kappa, \qquad q =: \delta\kappa \tag{21}$$

where ψ_κ and κ are both of the order $O(1)$ in the limit of small δ, the eigenvalue equation (9) yields

$$\psi_\kappa^2 + (f_{10} + f_{01}\kappa)\psi_\kappa + (g_{11}\kappa + g_{02}\kappa^2) + O(\delta) = 0 \quad . \tag{22}$$

In the vicinity of the bifurcation point $0 \leq \delta \ll 1$ one obtains the asymptotic relation [2]

$$\psi_\kappa^{(0/1)} \simeq -\frac{f_{10}}{2} - \frac{f_{01}}{2}\kappa \pm \sqrt{\left(\frac{f_{10}}{2} + \frac{f_{01}}{2}\kappa\right)^2 - g_{11}\kappa - g_{02}\kappa^2} \quad . \tag{23}$$

Here additionally the constraints $f_{01}^2 - 4g_{02} \geq 0$ and $(2f_{10}f_{01} - 4g_{11})^2 - f_{10}^2(f_{01}^2 - 4g_{02}) \geq 0$ on the expansion coefficients arise as a consequence of the real eigenvalues $\nu_q^{(0/1)}$. Changing to the original notation (cf. eq.(21)) one gets from eq.(23) the scaling relation

$$\nu_q^{(0/1)} - 1 = \delta\psi_\kappa^{(0/1)} \simeq \delta H_A^{(0/1)}\left(\frac{q}{\delta}\right) \tag{24}$$

where

$$H_A^{(0/1)}(x) := -\frac{f_{10}}{2} - \frac{f_{01}}{2}x \pm \sqrt{\left(\frac{f_{10}}{2} + \frac{f_{01}}{2}x\right)^2 - g_{11}x - g_{02}x^2} \quad . \tag{25}$$

[2] The $+$ sign corresponds to $\psi_\kappa^{(0)}$, as $\nu_q^{(0)} > \nu_q^{(1)}$.

The scaling relations for the characteristic function $\phi(q)$ and the lowest damping rate $\gamma_q^{(1)}$ read by inspecting eqs.(4) and (5)

$$\begin{aligned} \phi(q) &\simeq \delta H_A^{(0)}\left(\frac{q}{\delta}\right) \\ \gamma_q^{(1)} &\simeq \delta\left\{H_A^{(0)}\left(\frac{q}{\delta}\right) - H_A^{(1)}\left(\frac{q}{\delta}\right)\right\} \quad . \end{aligned} \tag{26}$$

Obviously a slowing down in the damping rate is observed in the limit $\delta \downarrow 0$, indicating the phase transition at $q = q_0 = 0$.

4 Case B: $q_0 \neq 0$

In this case the coefficient g_{10} is restricted by the relation (17). In the generic case however this relation reduces to the stronger condition

$$g_{10} < 0 \tag{27}$$

because a vanishing coefficient would require an additional constraint which can be removed by a small perturbation. Even if the situation $q_0 \neq 0$, $g_{10} = 0$ cannot be excluded, we will omit the discussion of this non generic case in this letter. Now eq.(11) reads

$$\begin{aligned} f(\delta, q) &= f_{01}(q - q_0) + O(\delta, (q - q_0)^2) \\ g(\delta, q) &= g_{10}\delta + g_{02}(q - q_0)^2 + O(\delta^2, \delta(q - q_0), (q - q_0)^3) \quad . \end{aligned} \tag{28}$$

Introducing a different scaling abbreviation due to the fact that $g(\delta, q)$ contains a contribution of first order

$$\frac{\nu_q}{\nu_c} - 1 =: \sqrt{\delta}\psi_\kappa, \qquad q - q_0 =: \sqrt{\delta}\kappa \tag{29}$$

eq.(9) leads to

$$\psi_\kappa^2 + f_{01}\kappa\psi_\kappa + g_{10} + g_{02}\kappa^2 + O(\sqrt{\delta}) = 0 \quad . \tag{30}$$

The following asymptotic behaviour in the vicinity of the bifurcation point holds

$$\psi_\kappa^{(0/1)} \simeq -\frac{f_{01}}{2}\kappa \pm \sqrt{\left(\left(\frac{f_{01}}{2}\right)^2 - g_{02}\right)\kappa^2 - g_{10}} \quad . \tag{31}$$

Due to the real valueness of $\psi^{(0/1)}$ the constraint $f_{01}^2 - 4g_{02} \geq 0$ is imposed on the expansion coefficients. Rewriting eq.(31) for the original quantities one obtains the scaling result

$$\frac{\nu_q^{(0/1)}}{\nu_c} - 1 = \sqrt{\delta}\psi_\kappa^{(0/1)} \simeq \sqrt{\delta}H_B^{(0/1)}\left(\frac{q - q_0}{\sqrt{\delta}}\right) \tag{32}$$

where

$$H_B^{(0/1)}(x) := -\frac{f_{01}}{2}x \pm \sqrt{\left(\left(\frac{f_{01}}{2}\right)^2 - g_{02}\right)x^2 - g_{10}} \quad . \tag{33}$$

Again the scaling relations for the characteristic function and the damping rate are immediately obtained from eqs.(4) and (5)

$$\begin{aligned} \phi(q) &\simeq \ln \nu_c + \sqrt{\delta} H_B^{(0)}\left(\frac{q-q_0}{\sqrt{\delta}}\right) \\ \gamma_q^{(1)} &\simeq \sqrt{\delta}\left\{ H_B^{(0)}\left(\frac{q-q_0}{\sqrt{\delta}}\right) - H_B^{(1)}\left(\frac{q-q_0}{\sqrt{\delta}}\right)\right\} \quad . \end{aligned} \tag{34}$$

Also in this case the damping rate shows of course a slowing down.

5 Examples

To illustrate the above mentioned general expansion this section contains the discussion of two simple one dimensional maps. The explicit expansion of the characteristic equation can be carried out easily and the meaning of the above mentioned two cases is illustrated.

To be definit consider the following two maps defined on the interval $[-1, 1]$ (Fig.2)

$$\begin{aligned} T_A(x) &= \left\{ \begin{array}{cl} -(2+r\epsilon)(x+\frac{1}{2})-1, & x \in [-1.-\frac{1}{2}] \\ 2x, & x \in [-\frac{1}{2}, \frac{1}{2}] \\ -(2+\epsilon)(x-\frac{1}{2})+1, & x \in [\frac{1}{2}, 1] \end{array} \right. \\ T_B(x) &= \left\{ \begin{array}{cl} -4(x+\frac{3}{4}), & x \in [-1, -\frac{1}{2}] \\ 2x, & x \in [-\frac{1}{2}, \frac{1}{2}] \\ -(2+\epsilon)(x-\frac{1}{2})+1, & x \in [\frac{1}{2}, 1] \end{array} \right. \quad . \end{aligned} \tag{35}$$

ϵ acts as a bifurcation parameter and the bifurcation takes place at $\epsilon = 0$ in both cases. For the map T_A two attracting sets exists for $\epsilon < 0$ in the intervals $[-1, 0]$ and $[0, 1]$. They merge into one attractor at the bifurcation point. In the system T_B an attractor existing in the region $[0, 1]$ for $\epsilon < 0$ collides with a repeller residing in the interval $[-1, 0]$. Both bifurcations should be analysed with respect to the fluctuating variable

$$u(x) = \left\{ \begin{array}{ll} k, & x \in [-1, 0) \\ k', & x \in (0, 1] \end{array} \right. \quad . \tag{36}$$

Using the results of [11] one can approximate the transfer operator (2) of both maps in the vicinity of the bifurcation point by the transfer matrix of a two state stochastic model, as far as the largest eigenvalues are concerned.

$$\left(\begin{array}{cc} (1-\delta')\exp(k'q) & \delta \exp(kq) \\ \delta' \exp(k'q) & (1-\delta)\exp(kq) \end{array} \right) \tag{37}$$

Here δ and δ' denote the transition rates from $[0, 1]$ to $[-1, 0]$ and from $[-1, 0]$ to $[0, 1]$ respectively. In the case A $\delta \sim \epsilon$, $\delta' \sim r\epsilon$ so that both rates vanish at the bifurcation point whereas in case B $\delta \sim \epsilon$ and δ' is finite in the limit $\epsilon \downarrow 0$. Without loss of generality we can restrict the following discussion to the special choice $k' = -1, k = 1$. This situation can be achieved in general by extracting the factor $\exp(q(k+k')/2)$ from eq.(37) and rescaling of the q axis, $q(k-k')/2 \to q$.

Case A, $\delta' = r\delta, \delta \downarrow 0$: In the limit $\delta \downarrow 0$ the eigenvalues of eq.(37) read $\exp(q), \exp(-q)$. Their degeneracy yields the phase transition point

$$q_0 = 0, \qquad \nu_c = 1 \quad . \tag{38}$$

The characteristic equation reads

$$\begin{aligned}(\nu_q - 1)^2 + \{2 - (1-\delta)e^q - (1-r\delta)e^{-q}\}(\nu_q - 1) \\ +2 - (1-\delta)e^q - (1-r\delta)e^{-q} - \delta - r\delta &= 0 \quad .\end{aligned} \tag{39}$$

Comparing eqs.(9) and (39) one obtains by making reference to the expansion (11)

$$\begin{aligned} f_{10} &= 1 + r \qquad & f_{01} &= 0 \\ g_{11} &= 1 - r \qquad & g_{02} &= -1 \end{aligned} \tag{40}$$

so that the scaling function (25) reads in this case

$$H_A^{(0/1)}(x) = -\frac{1+r}{2} \pm \sqrt{\left(\frac{1+r}{2}\right)^2 - (1-r)x + x^2} \quad . \tag{41}$$

Choosing $r = 1$ one recovers the well known result of the symmetry breaking chaos transition [8].

Case B, $\delta \downarrow 0, \delta'$ *is finite*: In the limit $\delta \downarrow 0$ the eigenvalues of eq.(37) read $\exp(q), (1-\delta')\exp(-q)$ and yield the phase transition point

$$q_0 = \frac{1}{2}\ln(1-\delta'), \qquad \nu_c = \sqrt{1-\delta'} \quad . \tag{42}$$

From eq.(37) one gets the eigenvalue equation

$$\begin{aligned}\left(\frac{\nu_q}{\nu_c} - 1\right)^2 + \{2 - (1-\delta)e^{q-q_0} - e^{-(q-q_0)}\}\left(\frac{\nu_q}{\nu_c} - 1\right) \\ +2 - (1-\delta)e^{q-q_0} - e^{-(q-q_0)} - \frac{\delta}{1-\delta'} &= 0 \quad .\end{aligned} \tag{43}$$

In comparison with eqs.(9) and (11) the following expressions for the expansion coefficients are obtained

$$f_{01} = 0 \qquad g_{10} = -\frac{\delta'}{1-\delta'} < 0 \qquad g_{02} = -1 \quad . \tag{44}$$

This yields with eq.(33) the scaling function

$$H_B^{(0/1)}(x) = \pm\sqrt{x^2 + \frac{\delta'}{1-\delta'}} \quad . \tag{45}$$

6 Conclusion

In this letter the scaling behaviour of thermodynamical quantities in the vicinity of a bifurcation point has been considered from a rather general point of view. Our approach uses only some basic assumptions concerning the degeneracy of eigenvalues of the transfer operator (2) which causes the phase transition. A straightforward expansion of the characteristic equation has lead us to explicit expressions determing the scaling behaviour. We want to stress that no reference to any special dynamical system has been used. By this fact the scaling behaviour is an universal property of the phase transition under consideration and is exclusively based on the eigenvalue structure of the transfer operator. Furthermore our approach explains that these scaling relations have been observed in rather different dynamical systems [9].

In this letter we have concentrated on a phase transition that is achieved by a doubly degenerated eigenvalue of the transfer operator. It is obvious that our approach can be extended to more complicated situations. Furthermore the influence of symmetries that are shared by special systems can be analysed. These topics will be investigated in a forthcoming publication.

7 Acknowledgement

One of the authors (W.J.) is greatly indebted to the Japan Society for the Promotion of Science for financial support by the postdoctoral fellowship for foreign researchers and to the Alexander von Humboldt Stiftung for financial support by the Feodor Lynen program.

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Figure captions:

Fig.1: Diagramatic view of the eigenvalues of the transfer operator (largest eigenvalue, second eigenvalue, remaining part of the spectrum (absolute value)). q_0 denotes the transition point and ν_c the corresponding eigenvalue.

Fig.2: Sketch of the maps eq.(35).