On Symbolic Dynamics of Space–Time Chaotic Models

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1 Introduction and Overview

The foundations of Statistical Mechanics rest to some extent on dynamical theories, as far as questions like relaxation and the approach to thermodynamic equilibrium as well as the justification of canonical distributions are concerned. Thus it is obvious that strong links between Statistical Physics and Nonlinear Dynamics exist. While from the traditional perspective the Hamiltonian character of the microscopic equations of motion and the thermodynamic limit play a crucial role we will sketch in this tutorial that the links between both fields extend to more general dynamical systems, e.g. even systems with dissipation or with few degrees of freedom.

During the last decades Nonlinear Dynamics and chaotic motion has attracted much interest from the principle and the experimental point of view and a huge amount of textbooks addressing different aspects are available (e.g. [1, 2, 3]). Contrary to the integrable case an individual trajectory is not a relevant object for chaotic motion since sensitivity with respect to the initial condition prevents in reality the reproduction of a particular trajectory. Thus it is obvious that statistical approaches become relevant. In particular, concepts developed in Statistical Physics for the treatment of many particle systems can be applied fruitfully to understand the motion in general dynamical systems. It is the scope of the present chapter to demonstrate such a link between Nonlinear Dynamics and (equilibrium) Statistical Physics with very elementary tools. The tutorial presented here just requires material which is usually contained in undergraduate physics or mathematics courses. Of course, much more general and sometimes quite elaborate presentations can be found in the literature (e.g. [4, 5, 6]) and the reader may find such references useful for further studies.

For our purpose we consider the simplest type of dynamical system, i.e. time discrete maps. Such a choice is quite common in Nonlinear Dynamics as long as principle and universal features are concerned in order to avoid all the technicalities which are related with real physical models. As a particular benefit we will be able to perform all our calculations explicitly by analytical means without imposing any approximation[1].

All concepts which establish a link between dynamics and statistics rely on a coarse grained description of the motion. Such approaches which are summarised in the term symbolic dynamics use suitable partitions of the full phase space. The trajectories are labelled by symbol sequences which tell in which order the different parts of the phase space are visited. It is one of the key observations that such a coarse graining does not reduce the amount of information provided one chooses a suitable partition. We will illustrate this fact in detail in section 2. The power of such a method goes in fact far beyond the simple examples we are addressing here (cf. e.g. [7, 8]) and one may use such concepts to investigate the topological complexity of the orbits in quite different settings.

An important characteristic of a dynamical system is given by the relative probability how often a "typical" trajectory visits certain parts of the phase space. Knowing such probabilities one may express time averages by phase space averages. In a more specific sense these questions are addressed by ergodic theory (cf. [9, 10, 6]). For the examples studied here we are able to answer such a question, i.e. closed formulas for the probabilities can be written down. Thus one can associate a Markov model with the deterministic equations of motion which describe the full motion from the statistical point of view. In some respect such a result provides the simplest explanation how one can relate deterministic dynamics with stochastic equations of motion and how irreversibility can be derived from time reversible dynamics [11]. In addition, the probabilistic approach tells us how frequently a finite symbol string appears in the symbolic dynamics mentioned above. Such a probability can be formally written as a Boltzmann weight of a spin chain and thus all dynamical mean values are rewritten in terms of a canonical equilibrium model. We will develop the details of such a statistical mechanics of dynamical systems in section 3. Up to that stage the correspondence between dynamical systems and Statistical Physics is more a formal trick and gives no essential new insight apart from the fact that the dependence of mean values on system parameters can be understood on a quite general level.

Qualitatively new phenomena show up when we consider dynamical models with many degrees of freedom, e.g. spatially extended dynamical systems. It is already a severe problem to carry over the notion of temporal chaos to the spatio–temporal case and to establish the concept of spatio–temporal chaos on some rigorous basis. Lacking a satisfactory definition which takes the spatial character of the degrees of freedom appropriately into account one may understand spatio–temporal chaos as states where spatio–temporal correlations decay exponentially [12, 13, 14]. There are alternative concepts available in the literature for characterising spatio–temporal chaos [15, 16, 17, 18], but

[1] Our presentation is formally exact but not rigorous, since we skip all the mathematically delicate questions.

most of them depend on some details of the underlying system. We will illustrate in section 4 how the above mentioned statistical description will shed some light on such a situation and how one can understand that systems of infinite extent behave differently compared to systems with a finite number of degrees of freedom. On a phenomenological level such a difference is already visible by considering the transient dynamics of dynamical models. Frequently one observes transient times which increase exponentially with the system size [19, 20] so that the model with a large number of degrees of freedom may never reach its stationary state. Our considerations will show that such a behaviour may be understood as a proper equilibrium phase transition appearing on the level of symbolic dynamics in the corresponding equilibrium spin system. Thus such a type of instability is intimately related to the limit of infinite system size and may be termed a phase transition in the proper sense (cf. [21]).

2 Piecewise Linear Markov Maps

We will illustrate the power of symbolic dynamics with a simple example, a time discrete dynamical system. In order to keep our discussion elementary and entirely analytic we restrict ourselves to one–dimensional maps and investigate the motion of the map depicted in figure 1. We do not need the corresponding analytical expression but of course it can be easily written down as

$$x_{n+1} = T(x_n) = \begin{cases} 2x + 1/3 & \text{if } 0 \le x \le 1/3 \\ -3/2(x-1) & \text{if } 1/3 < x \le 1 \end{cases} \quad . \tag{1}$$

The domain of the map splits in a natural way in two parts U_σ, $\sigma \in \{0, 1\}$, where $U_0 = [0, 1/3]$ and $U_1 = [1/3, 1]$ denote (closed) intervals. Roughly speaking we may associate to any initial condition x_0 an infinite sequence of symbols $(\sigma_0\sigma_1\sigma_2 \ldots)$ where σ_n tells us the interval which contains the trajectory at time n, $x_n \in U_{\sigma_n}$. Such a construction can be performed regardless of the particular partition $\{U_\sigma\}$ provided we discard for a moment the endpoints of the intervals where the symbols are not uniquely assigned[2]. The representation of x_0 in terms of its symbol sequence $(\sigma_0\sigma_1\sigma_2 \ldots)$ has a nice property with respect to the dynamics. Since $(\sigma_n\sigma_{n+1} \ldots)$ and $(\sigma_{n+1}\sigma_{n+2} \ldots)$ are the sequences corresponding to the phase space points x_n and x_{n+1} respectively, the dynamics induced by the map, $x_{n+1} = T(x_n)$ just boils down to a simple shift in symbol sequences. Without further precaution it is of course completely unclear whether the prescription $x \mapsto (\sigma_0\sigma_1 \ldots)$ is one to one and whether we loose some information when we describe the motion in terms of symbol sequences.

[2] Such a lack of uniqueness is also well known from the binary representation of real numbers.

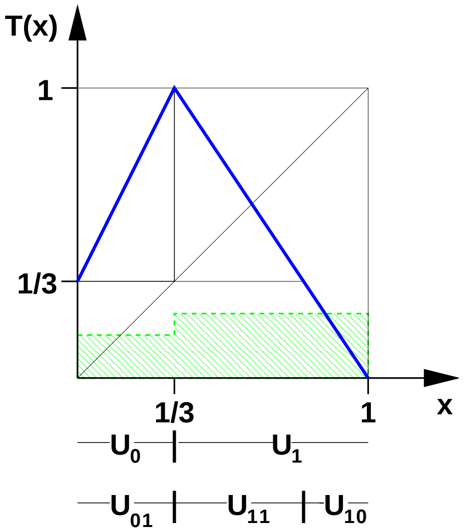

Fig. 1. Piecewise linear Markov map $T : [0,1] \to [0,1]$ (blue). The Markov partition $\{U_0, U_1\}$ and the first generation of cylinder sets $U_{\sigma\tilde{\sigma}}$ is indicated (cf. eq. (4)). Dashed line and hatched area represent the invariant density (green) (cf. eq. (8)).

To obtain the uniqueness of the labelling one needs special properties of the partition $\{U_\sigma\}$. From figure 1 it is obvious that T maps U_σ onto (a union of) $U_{\tilde{\sigma}}$ in a monotonic way. We can summarise such a behaviour by

$$T : \quad \begin{array}{l} U_0 \to U_1 \\ U_1 \to U_0 \cup U_1 \end{array} \quad . \tag{2}$$

Thus (when we disregard endpoints) transitions from U_0 to U_1 and transitions from U_1 to U_0 and U_1 are permitted whereas the transition from U_0 to U_0 is prohibited. We may condense these relations in a transition matrix $A_{\sigma\tilde{\sigma}}$ such that the matrix element is one when the transition $U_\sigma \to U_{\tilde{\sigma}}$ is permitted and zero otherwise. For our particular example the transition matrix reads

$$\underline{\underline{A}} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad . \tag{3}$$

If we consider symbol sequences $(\sigma_0\sigma_1\ldots)$ which contain only permitted nearest neighbour transitions $\sigma_n \to \sigma_{n+1}$ then we are indeed able to show that our construction $x \mapsto (\sigma_0\sigma_1\ldots)$ can be inverted, i.e. there is only one initial condition x_0 that generates the symbol sequence $(\sigma_0\sigma_1\ldots)$. To achieve such a goal we introduce the so called cylinder sets. Consider a finite symbol string $\sigma_0\sigma_1\ldots\sigma_{n-1}$ of n symbols which contains no forbidden transition. Then the set

$$U_{\sigma_0\ldots\sigma_{n-1}} := \{x \,|\, T^k(x) \in U_{\sigma_k},\, 0 \le k \le n-1\} \tag{4}$$

contains all initial conditions of trajectories which travel through the intervals U_{σ_k} during the finite time span $0 \le k \le n-1$. It is easy to check that eq. (4) yields a finite interval [3]. Obviously the cylinder sets obey the dynamical rule

$$T(U_{\sigma_0\ldots\sigma_{n-1}}) = U_{\sigma_1\ldots\sigma_{n-1}} \quad . \tag{5}$$

Since the map T is linear and in particular expansive on U_σ, i.e. has slope of modulus larger than one, the intervals (4) decrease exponentially in size if we consider longer symbol strings. In addition, $U_{\sigma_0\ldots\sigma_n} \subseteq U_{\sigma_0\ldots\sigma_{n-1}}$ holds. Thus eq. (4) yields a sequence of nested intervals which single out one phase space point x in the limit $n \to \infty$. That particular initial condition generates by construction the infinite symbol sequence $(\sigma_0\sigma_1\ldots)$. Hence we can invert our construction $x \mapsto (\sigma_0\sigma_1\ldots)$ as long as we confine ourself to admissible symbol sequences[4]. In particular, no information is lost when we change from the phase space to the coarse grained description in terms of symbol sequences.

We have developed a complete topological description of the dynamics of eq. (1) in terms of symbol sequences and the shift operation. The symbolic dynamics is completely determined by the transition matrix (3), i.e. by nearest neighbour transitions. Thus the selection rule for admissible symbol sequences, the so called grammar, is quite simple. Since just a nearest neighbour memory is involved we have obtained a (topological) Markov chain or a subshift of finite type. There is a simple way to represent such a process. Admissible symbol sequences may be generated by following the paths in the finite directed graph depicted in figure 2. Each path on that graph generates a symbol sequence according to the sites which are visited by the path. Such paths represent the symbol chain as well as the dynamics of the underlying deterministic map.

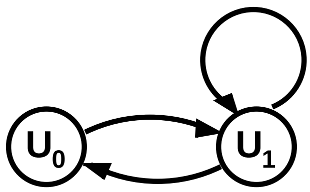

Fig. 2. Directed graph which generates the topological Markov chain of the dynamical system depicted in figure 1 (cf. eq. (2)).

[3] For forbidden symbol strings the set is either empty or consists just of isolated points.

[4] $(\sigma_0\sigma_1\ldots) \mapsto x$ is not injective. That property reflects the lack of uniqueness for phase space points which are mapped on the boundaries of U_σ. Thus, one has to match certain symbol sequences like for the binary representation of real numbers.

The nice description of the dynamics in terms of symbol sequences rests on the proper choice of the partition. As the previous example illustrates there are two essential constraints which the partition $\{U_\sigma\}$ has to fulfil. The map T must be expansive on each set U_σ and it must map boundary points of U_σ on boundary points. Expansiveness guarantees that the description in terms of symbol sequences becomes one to one. The condition on the boundary points ensures that we end up with the simple Markov property. Therefore one calls partitions which obey both constraints Markov partitions. If one of these constraints is relaxed the situation may become much more elaborate. If the constraint on the boundary points is dropped one still might have a generating partition [9] but the rules for the admissible symbol sequences may contain a long memory. If the expansiveness is violated then the uniqueness between state phase and symbol space may be lost but it is still possible to obtain useful information about the dynamics [7]. Here we just stick to the setting with the Markov property.

So far we have discussed the topological properties of the dynamics, i.e. how to characterise trajectories of the system. We have not addressed the problem "how frequently" a particular symbol sequence occurs. Such a topic is part of ergodic theory (cf. [9]). We do not give a full account here but just describe some basic features in elementary terms. Consider a "typical" orbit (i.e. an orbit emerging from initial conditions of a suitable set of finite Lebesgue measure). Fix a finite symbol string $\sigma_0 \ldots \sigma_{n-1}$ and compute the relative frequency how often the string occurs in the symbol sequence which is generated by the orbit. The relative frequency coincides with the probability that the orbit visits the cylinder set (4). If we denote this probability by $\mu(U_{\sigma_0 \ldots \sigma_{n-1}})$, observe the dynamical law (5), and require the stationarity of the probabilities then the balance condition of the probabilities in each time step tells us that

$$\mu(U_{\sigma_1 \ldots \sigma_{n-1}}) = {\sum_{\sigma_0}}' \mu(U_{\sigma_0 \ldots \sigma_{n-1}}) = \sum_{\sigma_0} A_{\sigma_0 \sigma_1} \mu(U_{\sigma_0 \ldots \sigma_{n-1}}) \quad . \tag{6}$$

Here $\sum'$ denotes the sum with respect to all admissible symbol sequences. Of course it can be rewritten using the transition matrix (3). There is a second trivial constraint the probabilities have to obey. Recalling the definition of cylinder sets (4) it is obvious that skipping the last symbol we obtain cylinder sets of the previous generation, i.e. $\cup'_{\sigma_{n-1}} U_{\sigma_0 \ldots \sigma_{n-1}} = U_{\sigma_0 \ldots \sigma_{n-2}}$. Thus the sum rule

$$\mu(U_{\sigma_0 \ldots \sigma_{n-2}}) = {\sum_{\sigma_{n-1}}}' \mu(U_{\sigma_0 \ldots \sigma_{n-1}}) = \sum_{\sigma_{n-1}} \mu(U_{\sigma_0 \ldots \sigma_{n-1}}) A_{\sigma_{n-2} \sigma_{n-1}} \tag{7}$$

has to be fulfilled.

Equation (6) is nothing else but the famous Frobenius–Perron equation (cf. e.g. [10]). It admits many different solutions. We are here interested in typical orbits and in that case the probabilities scale with the local expansion rates which the orbit points visit through its itinerary. If $\gamma_\sigma := |T'(x)|$, $x \in U_\sigma$ denotes the modulus of the slope of the map on U_σ we make the ansatz[5]

$$\mu(U_{\sigma_0 \ldots \sigma_{n-1}}) = h_{\sigma_0} \frac{1}{\gamma_{\sigma_0}} \times \ldots \times \frac{1}{\gamma_{\sigma_{n-2}}} \nu_{\sigma_{n-1}} \quad . \tag{8}$$

The reason for such an ansatz is quite simple (although the mathematical proof has been a challenge from the rigorous point of view). Since the map expands on each time step by a factor γ_{σ_k} the geometric length of the cylinder set (4) scales as $(\gamma_{\sigma_0} \ldots \gamma_{\sigma_{n-1}})^{-1}$. Thus the probabilities according to eq. (8) yield a continuous probability density[6]. In the first and the last symbol the expression differs from local expansion rates in order to take care of the two conditions (6) and (7). In fact, plugging the ansatz (8) into eq. (7) we just get

$$\nu_\sigma = \sum_{\tilde{\sigma}} \frac{A_{\sigma\tilde{\sigma}}}{\gamma_\sigma} \nu_{\tilde{\sigma}} \tag{9}$$

and the matrix appearing on the right hand side is usually called the transfer matrix of the system. The other boundary contribution is obtained from eq. (6) as

$$h_\sigma = \sum_{\tilde{\sigma}} h_{\tilde{\sigma}} \frac{A_{\tilde{\sigma}\sigma}}{\gamma_{\tilde{\sigma}}} \tag{10}$$

which is just the adjoint equation of the expression (9). Finally normalisation of the probabilities results in

$$1 = \sum_\sigma \mu(U_\sigma) = \sum_\sigma h_\sigma \nu_\sigma \quad . \tag{11}$$

Since the probabilities obey by construction the condition of stationarity (cf. eq. (6) the corresponding probability density is usually called the invariant measure (actually the SRB measure). For our particular example eq. (1) we obviously have $\gamma_0 = 2$, $\gamma_1 = 3/2$, $(\nu_0, \nu_1) = (1/4, 1/2)$, and $(h_0, h_1) = (1, 3/2)$. Thus $\mu(U_0) = 1/4$, $\mu(U_1) = 3/4$ and the corresponding probability density (cf. figure 1) is constant on U_σ.

With the probabilities (8) we are able to compute average values of phase space functions. According to ergodic theory these averages may also be understood as long time averages. For analytical computations evaluation of phase space averages using eq. (8) is of course simpler. In order to avoid some technicalities we restrict to phase space functions $h(x)$ which are constant on cylinder sets (4). The reader may consult e.g. [23] for the treatment

[5] Other measures, which can be obtained by different scaling factors, correspond to non typical orbits. These measures are useful for the multifractal analysis of dynamical systems [4, 22].

[6] Ergodic theory then tells us that these probabilities are generated by (Lebesgue) almost all initial conditions.

of more general cases, which however do not pose essential problems. Assume that for some fixed value N the phase space function $h(x)$ is constant on $U_{\sigma_0\ldots\sigma_{N-1}}$ and let $h_{\sigma_0\ldots\sigma_{N-1}}$ denote the corresponding value of $h(x)$. Then the average value is given by

$$\langle h(x)\rangle = \sum_{\sigma_0,\ldots,\sigma_{N-1}} h_{\sigma_0\ldots\sigma_{N-1}}\mu(U_{\sigma_0\ldots\sigma_{N-1}}) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} h(x_k) \quad . \tag{12}$$

Taking the relation between the temporal dynamics and the symbol shift into account (cf. eq. (5)) correlation functions may be evaluated too, e.g.

$$\langle h(T^k(x))h(x)\rangle = \sum_{\sigma_0,\ldots,\sigma_{N+k-1}} h_{\sigma_k\ldots\sigma_{N+k-1}} h_{\sigma_0\ldots\sigma_{N-1}}\mu(U_{\sigma_0\ldots\sigma_{N+k-1}}) \quad . \tag{13}$$

Thus for piecewise linear Markov maps we are able to compute all dynamical features by analytical methods. In that sense linear Markov maps are a suitable testing ground for problems in Nonlinear Dynamics (cf. e.g. [24]). The scheme that has been sketched in the present section is of course not restricted to the simple example eq. (1). It works for general Markov maps. The notion of a Markov map is quite flexible and allows the modelling of quite complex situations, e.g. intermittent motion [25, 26, 27]. In such cases the complexity of the motion is represented by a richer alphabet of symbols from which the symbolic dynamics is derived and sometimes by a more complicated grammar.

3 Statistical Mechanics of Simple Maps

The symbolic description developed in section 2 permits to link dynamical properties with equilibrium statistical mechanics on a formal level. If we inspect the dynamical averages (12) or (13) then these quantities can be rewritten as canonical mean values of an observable $h_{\sigma_0\ldots\sigma_{N-1}}$ in spin chain provided we identify the probability with a Boltzmann weight and define the spin Hamiltonian according to

$$\exp(-H_{\sigma_0\ldots\sigma_{n-1}}) := \mu(U_{\sigma_0\ldots\sigma_{n-1}}) \quad . \tag{14}$$

At the moment such a formulation is just a formal trick. Nevertheless we see that dynamical averages are equivalent to the thermostatics of a spin chain $\sigma_0\ldots\sigma_{n-1}$ with Hamiltonian being given by eq. (14). The lattice translation in the spin chain corresponds to the temporal evolution of the dynamical system (cf. eq. (13) or eq. (5)) and the thermodynamic limit $n\to\infty$ yields a finer resolution by cylinder sets (4), i.e. the long time limit. As for the structure of the Hamiltonian eq. (8) tells us that the quantity for piecewise linear Markov maps reads

$$H_{\sigma_0\ldots\sigma_{n-1}} = \sum_{k=0}^{n-1} \ln \gamma_{\sigma_k} - \ln h_{\sigma_0} - \ln(\nu_{\sigma_{n-1}}/\gamma_{\sigma_{n-1}}) \quad . \tag{15}$$

The Hamiltonian describes a model consisting of two level systems, where the single site quantity $\sigma_k = 0, 1$ indicates which level at site k is occupied. Using the notation of lattice gases the index σ_k indicates whether the lattice site is occupied by a particle, whereas if we identify the index σ_k with the quantum number of a spin then the Hamiltonian (15) describes a spin chain. Here we adopt the latter interpretation. Thus apart from boundary contributions at $k = 0$ and $k = n - 1$ the Hamiltonian describes a spin system without interaction but subjected to a magnetic field of strength $\ln(\gamma_1/\gamma_0)$[7]. Of course one has to take into account that only spin states according to the transition matrix (cf. eq. (3)) are permitted. Such a constraint can be viewed as an infinitely strong hard core repulsion between neighbouring lattice sites. Nevertheless common wisdom of equilibrium statistical mechanics tells us that no phase transition occurs, i.e. that the mean values (12) depend analytically on the system parameters, that is on the local slopes of the map. Furthermore correlations (13) decay exponentially.

One should note that in the just mentioned formulation there does not exist any temperature. Since the statistical approach just reproduces dynamical averages, there is nothing like the zero law of thermodynamics available. Nevertheless one may introduce a temperature as a formal parameter.Then different kinds of invariant measures are singled out which can be fruitfully used to analyse multifractal properties and the statistics of fluctuations of finite time averages. For such advanced topics we refer the interested reader to the literature [28, 29, 30, 4].

So far we have restricted the discussion to piecewise linear maps. In order to study the influence of a finite curvature we analyse a simple example. Let us consider the one–dimensional map shown in figure 3. A Markov partition consisting of two sets $U_\pm$ is indicated. On these sets the map has a simple curvature, since the slope takes two different values. It will simplify the subsequent analysis considerably that on a finer partition the map is again a piecewise linear Markov map. For our particular example not all initial conditions stay in the domain of the map. On iteration a Cantor set is singled out which contains all points that remain in the domain forever. That is easily seen if we compute successive cylinder sets. As indicated in figure 3 the cylinder sets yield the usual construction of a simple multifractal[8].

If we restrict the analysis to the invariant set then most considerations of the previous section can be applied. For our map all transitions $U_\sigma \to U_{\tilde{\sigma}}$ are possible. Thus no nontrivial transition matrix appears and all symbol

[7] Using $\ln \gamma_\sigma = \sigma \ln \gamma_1 + (1 - \sigma) \ln \gamma_0$ the Hamiltonian may be cast in the standard form for a spin system.

[8] The invariant set studied here is a chaotic saddle with nontrivial fractal dimension in contrast to attractors which have been studied in the previous section.

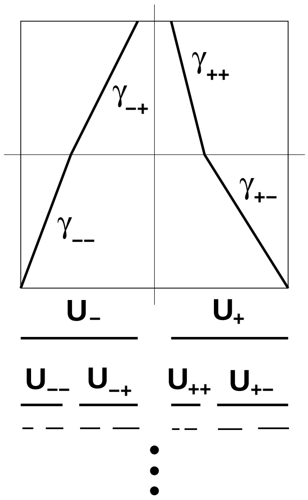

Fig. 3. One–dimensional Ising map. A Markov partition $\{U_+, U_-\}$ and the first generations of cylinder sets are indicated. Moduli of the slopes of the map are denoted by $\gamma_{\sigma\tilde{\sigma}}$.

sequences are admissible. For the probabilities a slight modification of eq. (8) applies since the slopes of the map are not constant on U_σ. But by construction the slopes are constant on the cylinder sets $U_{\sigma\tilde{\sigma}}$ of the first generation. Therefore, we apply essentially the same idea that lead us to eq. (8) for obtaining the invariant measure by using the partition given by $U_{\sigma\tilde{\sigma}}$. Thus the invariant measure is again determined by the local slopes and it reads

$$\mu(U_{\sigma_0\ldots\sigma_{n-1}}) = h_{\sigma_0\sigma_1}\frac{\alpha^{-1}}{\gamma_{\sigma_0\sigma_1}} \times \ldots \times \frac{\alpha^{-1}}{\gamma_{\sigma_{n-3}\sigma_{n-2}}}\nu_{\sigma_{n-2}\sigma_{n-1}} \quad . \tag{16}$$

Here the factor $\alpha < 1$ takes into account that we are dealing with a repelling set, i.e. on each iteration step a fraction $\alpha < 1$ of phase space points will leave the domain. As a consequence the escape factor α appears in eq. (16) (cf. e.g. [22] for a more extensive discussion of chaotic saddles and [31] for the relation between chaotic saddles, fractals, iterated function systems, and concepts of modern ergodic theory). Thus plugging eq. (16) into the constraint eq. (7) and observing that the transitions $U_{\sigma_0\sigma_1} \to U_{\sigma_1\sigma_2}$ between cylinder sets of the first generation are permitted we obtain the eigenvalue equation (cf. eq. (9) for the case of attracting sets)

$$\alpha\nu_{\sigma_0\sigma_1} = \sum_{\sigma_2}\frac{1}{\gamma_{\sigma_0\sigma_1}}\nu_{\sigma_1\sigma_2} \quad . \tag{17}$$

Accordingly the balance condition (6) yields the adjoint problem

$$\alpha h_{\sigma_1\sigma_2} = \sum_{\sigma_0} h_{\sigma_0\sigma_1}\frac{1}{\gamma_{\sigma_0\sigma_1}} \quad . \tag{18}$$

These eigenvalue equations determine the invariant probability as well as the escape factor[9].

Our considerations show that the dynamical properties of the map with curvature are again equivalent to a spin chain with Hamiltonian being given by

$$H_{\sigma_0 \ldots \sigma_{n-1}} \simeq \sum_{k=0}^{n-2} \ln \gamma_{\sigma_k \sigma_{k+1}} + n \ln \alpha \tag{19}$$

where the boundary contributions have not be written down explicitly. Thus we obtain an Ising model with nearest neighbour interaction. Written in conventional notation[10] the exchange constant is given by $\ln(\gamma_{--}/\gamma_{-+}) + \ln(\gamma_{++}/\gamma_{+-})$. Thus, it is just the curvature which causes a short range interaction in the corresponding spin Hamiltonian. In addition an asymmetry of the map is reflected in eq. (19) by an external field, which can be read off as $\ln(\gamma_{++}/\gamma_{--})$. Finally the escape factor enters the ground state energy of the Hamiltonian (19).

The features that have been deduced for the special example are quite typical. In fact, since expansiveness causes an exponential decrease of the length of cylinder sets a map with curvature will always result in a spin Hamiltonian with exponentially decreasing interactions, although such interactions may involve many spins. Thus no phase transition, i.e. a qualitative change of the dynamics, can be caused by curvatures of expanding maps. Such a property is relevant in different contexts too, i.e. for nice convergence properties of expansions in terms of periodic orbits [32].

We have illustrated that general expansive Markov maps result in one–dimensional spin Hamiltonians with short range interactions. Such systems do not display phase transitions, i.e. averages depend analytically on the model parameters and correlations decay exponentially. Roughly speaking temporal chaos corresponds to the paramagnetic or high temperature phase of the corresponding equilibrium model.

Qualitative changes in the dynamics (e.g. bifurcations) may occur if the dynamical system looses expansivity, i.e. when the dynamics becomes non-hyperbolic. Then long range interactions may be induced in the spin Hamiltonian, as was explicitly shown for intermittent dynamics [26]. The emerging phase transition displays the qualitatively different types of motion which are related with the particular bifurcation (cf. e.g. [27]). For the purpose of our tutorial we will stick to the simple case of expansive motion. We will demonstrate that in spatially extended systems a different mechanism which is intimately related to the limit of large system size may induce qualitative

[9] For the case of attracting sets the escape factor turns out to be $\alpha = 1$ by Perron's theorem.

[10] $\ln \gamma_{\sigma\tilde{\sigma}} = [(1+\sigma)(1+\tilde{\sigma}) \ln \gamma_{++} + (1+\sigma)(1-\tilde{\sigma}) \ln \gamma_{+-} + (1-\sigma)(1+\tilde{\sigma}) \ln \gamma_{-+} + (1-\sigma)(1-\tilde{\sigma}) \ln \gamma_{--}]/4$.

changes of the motion. Such a transition may be termed a phase transition in the proper sense (cf. [21]).

4 Spatially Extended Chaotic Models

To investigate the qualitatively new features which are related with a spatial degree of freedom we again resort to the analysis of a simple class of model systems. It has turned out that the study of coupled map lattices provides useful information [33] at least if one wants to understand dynamics from a qualitative and phenomenological point of view. Consider a spatially one–dimensional lattice of length L and assume that each lattice site ν, $0 \leq \nu \leq L-1$ is associated with a variable $x^{(\nu)}$. The full state of the system is described by $\underline{x} = (x^{(0)}, \ldots, x^{(L-1)})$. For the dynamics we suppose that on each lattice site there acts a map $f(x)$ and neighbouring lattice sites are coupled through a coupling function $\Phi : [-1,1]^L \rightarrow [-1,1]^L$. Then the whole time discrete motion is determined by $\underline{x}_{n+1} = T(\underline{x}_n)$ where[11]

$$\underline{x}_{n+1} = T(\underline{x}_n) = \Phi\left[f(x_n^{(0)}), f(x_n^{(1)}), \ldots, f(x_n^{(L-1)})\right] \ . \tag{20}$$

Boundary conditions will play no crucial role for our purpose, thus one may consider periodic boundary conditions. Although no microscopic derivation of such dynamical models has been performed from first principles models of the type eq. (20) had turned out quite useful for numerical studies [34]. Particular numerical investigations where the system size becomes important concern the exponential increase of transient motion [19], the study of nonequilibrium phase transitions, and the related critical behaviour [35, 36]. These investigations already indicate that in the limit of large L spatially extended models may display qualitative new features. Hence we will focus on the influence of the system size on the motion of the system.

In order to keep our discussion as simple as possible we will focus on a particular coupled map lattice [37]. For the single site map f we consider the system already introduced in section 3 (cf. figure 3). To get an idea how the dynamics will behave let us first consider the trivial case of uncoupled maps, where Φ is the identity map. At each lattice site just the map f acts on $x_n^{(\nu)}$ and induces its symbol sequence $(\sigma_0^{(\nu)}\sigma_1^{(\nu)}\ldots)$. Thus the motion of the whole state vector $\underline{x}_n$ is described by a two–dimensional symbol lattice

[11] A very popular choice for the coupling consists in diffusive coupling where

$$x_{n+1}^{(\nu)} = f(x_n^{(\nu)}) + \varepsilon\left[f(x_n^{(\nu+1)}) + f(x_n^{(\nu-1)}) - 2f(x_n^{(\nu)})\right]$$

i.e. $(\Phi[\underline{x}])^{(\nu)} = x^{(\nu)} + \varepsilon(x^{(\nu+1)} + x^{(\nu-1)} - 2x^{(\nu)})$. But we do not restrict ourselves to such a particular case.

$$\begin{array}{cccc} \sigma_0^{(0)} & \sigma_1^{(0)} & \sigma_2^{(0)} & \ldots \\ \sigma_0^{(1)} & \sigma_1^{(1)} & \sigma_2^{(1)} & \ldots \\ \vdots & \vdots & \vdots & \\ \sigma_0^{(L-1)} & \sigma_1^{(L-1)} & \sigma_2^{(L-1)} & \ldots \end{array} =: (\underline{\sigma}_0\underline{\sigma}_1\underline{\sigma}_2\ldots) \quad . \tag{21}$$

$\sigma_n^{(\nu)}$ tells us which interval $U_\pm$ is occupied by (the corresponding component of) the phase space point at time n on lattice site ν. The first dimension of such a spin lattice corresponds to the spatial extension of the dynamical system whereas the second dimension reflects the temporal motion of the original system. The shift operation in the lattice yields either the spatial translation or the temporal evolution, as in the case of nonextended systems (cf. section 2). Obviously the uncoupled map lattice is a piecewise linear Markov map, with phase space being given by $[-1,1]^L$.

For the introduction of probabilities we again consider the associated cylinder sets, which are related with the time evolution over a finite period $0 \leq k \leq n-1$ (cf. eq. (4)). Different cylinder sets are labelled by finite two–dimensional spin lattices $(\underline{\sigma}_0 \ldots \underline{\sigma}_{n-1})$. Since we consider the uncoupled case the probabilities are just the products of the quantities of the individual maps (cf. eq. (16)) and we thus have

$$H_{\underline{\sigma}_0\cdots\underline{\sigma}_{n-1}} := -\ln\mu(U_{\underline{\sigma}_0\cdots\underline{\sigma}_{n-1}}) \simeq \sum_{\nu=0}^{L-1}\sum_{k=0}^{n-2} \ln\gamma_{\sigma_k^{(\nu)}\sigma_{k+1}^{(\nu)}} + nL\ln\alpha \quad . \tag{22}$$

For simplicity of notation the boundary contributions have not been written down explicitly. Thus the corresponding spin Hamiltonian describing our dynamical system consists of a two–dimensional lattice where only a nearest neighbour intra–chain interaction occurs. That is of course not surprising since we started from a dynamical system without spatial interaction. One already expects that on turning on the spatial interaction we will obtain an associated spin system with real two–dimensional interaction. Such a scenario opens the possibility for a phase transition in the thermodynamic limit $L \to \infty$.

To prepare the formal steps that we need to investigate the coupled case let us first have a look at the phase space structure of the uncoupled map lattice. For the purpose of visualisation we concentrate on the case of two coupled maps $L = 2$, but the considerations go along the same lines for general L. The Markov partition of the uncoupled map lattice consists of sets which are just the direct product of the partition of the single site maps, $U_{\underline{\sigma}} = U_{\sigma^{(0)}} \otimes \ldots \otimes U_{\sigma^{(L-1)}}$. The full phase space $[-1,1]^L$ is divided by that Markov partition $\{U_{\underline{\sigma}}\}$ in 2^L parts (cf. figure 4). Cylinder sets of the first generation $U_{\underline{\sigma}\underline{\tilde{\sigma}}}$ divide the space in 2^{2L} parts and so on. Cylinder sets of the generation $n-1$ are labelled by finite symbol lattices of size $L \times n$. The lack of spatial coupling in the map lattice is reflected by the fact that the cylinder sets admit a product structure (i.e. the extension of neighbouring sets in the direction of the coordinate axis is equal). The uncoupled map lattice itself, which we denote by T_0, maps successive cylinder sets in a linear way (cf. eq. (5)), i.e.

$$T_0 : U_{\underline{\sigma\tilde{\sigma}}} \to U_{\underline{\tilde{\sigma}}} \tag{23}$$

is a linear map with Jacobian $\prod_{\nu=0}^{L-1} \gamma_{\sigma^{(\nu)}\tilde{\sigma}^{(\nu)}}$ when restricted to the set $U_{\underline{\sigma\tilde{\sigma}}}$.

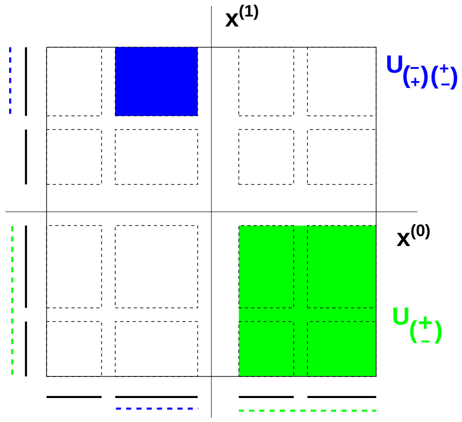

Fig. 4. Phase space structure of the uncoupled map lattice T_0 for $L = 2$. Cylinder sets of the first generation $U_{\underline{\sigma\tilde{\sigma}}}$ (dashed rectangles), and cylinder sets of the single site maps (full lines, cf. figure 3). Coloured areas show a cylinder set (blue) and its image (green), broken lines indicate the sets and its images for the single site maps.

The introduction of a coupling Φ will remove the just mentioned product structure of the cylinder sets. For an arbitrary choice of the coupling the Markov partition may become quite complicated. In order to keep the analysis as simple as possible we will define the coupling function in such a way that a simple pattern for the cylinder sets will remain so that no tedious analytical computations become necessary. Consider the cylinder sets $U_{\underline{\sigma\tilde{\sigma}}}$ depicted in figure 4. Now move the "inner vertex" of each set by some amount as shown in figure 5. If we require that the new cylinder sets, which we will denote by $V_{\underline{\sigma\tilde{\sigma}}}$, are still mapped in a linear fashion like in the uncoupled case (cf. eq. (23))

$$T : V_{\underline{\sigma\tilde{\sigma}}} \to U_{\underline{\tilde{\sigma}}} \tag{24}$$

we obtain a new dynamical system T which is now a coupled map lattice. Since this new model is still a piecewise linear Markov map we are able to solve the dynamics along the lines of section 2.

In what follows we do not need the analytical formula of the just defined map lattice T (cf. the relevance of eq. (1) for the considerations in section 2). Nevertheless we will supply the expression for completeness. The coupling

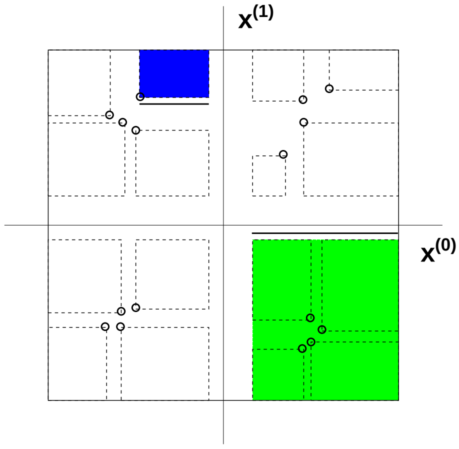

Fig. 5. Phase space structure of the coupled map lattice T for $L = 2$. Dashed rectangles: cylinder sets of the first generation $V_{\underline{\sigma\tilde{\sigma}}}$, open circles: inner vertexes of the cylinder sets. Coloured areas depict a cylinder set (blue) and its image (green) (cf. figure 4), full lines indicate an edge of a cylinder set and its image.

Φ is constructed in such a way that it maps the cylinder set $V_{\underline{\sigma\tilde{\sigma}}}$ of the full system (cf. figure 5) to the cylinder set $U_{\underline{\sigma\tilde{\sigma}}}$ of the map lattice without interaction (cf. figure 4). Then the dynamics is given by

$$x_{n+1}^{(\nu)} = f\left((\Phi[\underline{x}])^{(\nu)}\right) \quad . \tag{25}$$

In order to write down the analytical expression for Φ consider a cylinder set $V_{\underline{\sigma\tilde{\sigma}}}$ and denote its outer vertex, i.e. the common vertex of $V_{\underline{\sigma\tilde{\sigma}}}$ and $U_{\underline{\sigma\tilde{\sigma}}}$ by $\underline{a}_{\underline{\sigma\tilde{\sigma}}}$. Consider the edges of $V_{\underline{\sigma\tilde{\sigma}}}$ and of $U_{\underline{\sigma\tilde{\sigma}}}$ which are parallel to the ν–axis and denote the ratio of the length of these edges by $r_{\underline{\sigma\tilde{\sigma}}}^{(\nu)}$. Then the coupling function reads

$$(\Phi[\underline{x}])^{(\nu)} = r_{\underline{\sigma\tilde{\sigma}}}^{(\nu)}(x^{(\nu)} - a_{\underline{\sigma\tilde{\sigma}}}^{(\nu)}) + a_{\underline{\sigma\tilde{\sigma}}}^{(\nu)}, \qquad \underline{x} \in V_{\underline{\sigma\tilde{\sigma}}} \quad . \tag{26}$$

Spatial coupling is in particular mediated by the factor $r_{\underline{\sigma\tilde{\sigma}}}^{(\nu)}$ which may depend on symbols of neighbouring lattice sites. Thus the spatial coupling removes the product structure visible in figure 4. Finally, in order to convert our map lattice (25) to the form (20) one just introduces new coordinates $\underline{z} = \Phi(\underline{x})$ and applies the coupling Φ to both sides of eq. (25).

For the following analysis we will just need the Jacobian of the map lattice T. That quantity is of purely geometric origin, since it is determined by the volume ratio of $U_{\underline{\tilde{\sigma}}}$ and $V_{\underline{\sigma\tilde{\sigma}}}$ (cf. eq. (24) and figure 5). Let $t_{\underline{\sigma\tilde{\sigma}}}^{(\nu)}$ denote the ratio of the edges of $U_{\underline{\tilde{\sigma}}}$ and $V_{\underline{\sigma\tilde{\sigma}}}$ which are parallel to the ν–axis (cf. figure 5)[12]. Then, the Jacobian of the map T reads.

[12] $t_{\underline{\sigma\tilde{\sigma}}}^{(\nu)} = \gamma_{\sigma^{(\nu)}\tilde{\sigma}^{(\nu)}} r_{\underline{\sigma\tilde{\sigma}}}^{(\nu)}$.

$$|\mathrm{det} DT(\underline{x})| = \prod_{\nu=0}^{L-1} t^{(\nu)}_{\underline{\sigma}\underline{\tilde{\sigma}}} \qquad (\text{if } \underline{x} \in V_{\underline{\sigma}\underline{\tilde{\sigma}}}) \quad . \tag{27}$$

According to our general considerations of sections 2 and 3 these expansion rates determine the invariant measure and we end up with (cf. eq. (19))

$$H_{\underline{\sigma}_0 \cdots \underline{\sigma}_{n-1}} = -\ln \mu(V_{\underline{\sigma}_0 \cdots \underline{\sigma}_{n-1}}) \simeq \sum_{k=0}^{n-2} \sum_{\nu=0}^{L-1} \ln t^{(\nu)}_{\underline{\sigma}_k \underline{\sigma}_{k+1}} + nL \ln \alpha \tag{28}$$

where we have dropped the boundary terms, and α denotes the escape factor of our map lattice. Since α just determines the overall normalisation of the probabilities (i.e. the value of the ground state energy) we refrain from writing down the corresponding eigenvalue problem explicitly (cf. eq. (17) or (18)). The quantities $t^{(\nu)}_{\underline{\sigma}_k \underline{\sigma}_{k+1}}$ are the important parameters which enter the coupled map lattice and determine the coupling (cf. eq. (26)) as well as the invariant measure. In dynamical terms these parameters determine the local slope which the phase space point experiences at lattice site ν.

Depending on the particular choice of our coupling parameters $t^{(\nu)}_{\underline{\sigma}\underline{\tilde{\sigma}}}$ the map lattice T is equivalent to the thermostatics of a Hamiltonian on a two–dimensional lattice with two–dimensional interaction. If we choose for instance

$$t^{(\nu)}_{\underline{\sigma}\underline{\tilde{\sigma}}} = t^{(\nu)}_{\underline{\sigma}\underline{\tilde{\sigma}}}(h, J) := \exp[-h\sigma^{(\nu)} - J\sigma^{(\nu)}(\tilde{\sigma}^{(\nu)} + \sigma^{(\nu+1)}) + e_0] \tag{29}$$

then eq. (28) yields the famous nearest neighbour coupled Ising Hamiltonian. It is well known that the thermodynamic equilibrium state develops a phase transition at $h = 0$, $J = J_c = \mathrm{Artanh}(\sqrt{2} - 1)$ in the thermodynamic limit. In the high temperature phase $0 \leq J < J_c$ correlations decay exponentially whereas for $J > J_c$ two pure phases develop. These standard results now transfer to the dynamical system. In the regime $J < J_c$, i.e. when the quantities (29) do not depend strongly on neighbouring lattice sites, spatio–temporal correlations decay exponentially. Such a behaviour, sometimes called space–time chaos, is expected in the regime of weak spatial coupling. When the spatial coupling exceeds a critical value, then the invariant measure decomposes into two different ergodic components, which are not mixed by the dynamics and which correspond to the pure phases in the associated equilibrium system. If one adds a small asymmetry, i.e. a small field h, then the two limits $h \uparrow 0$ and $h \downarrow 0$ result in two different dynamical states where the magnetisation $\sum_{n,\nu} \sigma_n^{(\nu)}$ attains a finite value in the thermodynamic limit. In dynamical terms the magnetisation is identical to the dynamical mean value of $S_n = \sum_\nu \sigma_n^{(\nu)}$ (cf. eq. (12)). Thus the majority of single site maps stays in their left or right part of the domain although the single site dynamics looks like the graph in figure 3. To achieve such a phase transition a modulation of the local slopes by a factor of size ~ 4 is already sufficient, according to eq. (29). For finite system size L no phase transition occurs at all, long time averages depend analytically on system parameters, all correlations decay exponentially, and the system is mixing. Thus the behaviour described here is a indeed transition which is caused just by the limit of large system size.

The behaviour illustrated by our special model reflects typical features which can be found in quite general coupled map lattices. But the analysis is often very delicate. It has been shown rigorously that the weak coupling regime of a coupled map lattice can be mapped onto the high–temperature phase of a spin–lattice so that the dynamics displays a spatio–temporal mixing state, i.e. the motion generates exponentially decaying spatio–temporal correlations (cf. e.g. [12, 13, 14]) provided certain technical constraints on the single site dynamics and the coupling are fulfilled. Such a behaviour may be used for the definition of space–time chaos. But one has to admit that no fully satisfactory definition is available in the literature (cf. e.g. [15, 16, 38] for alternative concepts). The corresponding spin Hamiltonian is in general far more complicated as in our simple example since complicated anisotropic and many particle interactions appear. Thus the Statistical Mechanics of a typical coupled map lattice seems to resemble the description of spin glasses instead of simple ferromagnetic phase transitions. For larger coupling strength it is conjectured that the breakdown of the spatio–temporal chaotic state takes place through some phase transition, as in our simple example, although the nature of the transition in the typical case is still unclear (cf. e.g. [39, 40] for some explicit examples and [21] for some general considerations). There are of course alternatives for the onset of coherent pattern formation e.g. through dynamical bifurcations (cf. e.g. [18]) which are well known from the theory of low–dimensional dynamical systems. As such a mechanism is not related to the limit of large system size one should not call such a transition a phase transition in the proper sense.

So far we have emphasised how the invariant measure (i.e. the probabilities) are modified by the spatial coupling in the limit of infinite systems size, keeping the topological properties, i.e. the symbolic dynamics, rather simple. It is in fact less known, how the coupling may influence topological properties, and whether these effects are relevant in the thermodynamic limit. Case studies in simple situations (e.g. two–dimensional phase spaces) show already [41] that a quite complicated grammar may appear. In fact such questions have to be addressed in order to get a full overview of the dynamical behaviour in terms of associated spin systems.

Within our tutorial we have confined ourselves to the construction of symbolic dynamics through Markov partitions. There are of course alternatives for obtaining coarse grained descriptions of dynamical models (cf. e.g. [42] for the relation between extended dynamical systems and kinetic Ising models). Although there are a plenty of unanswered problems the elementary introduction presented here indicates, that the link between nonlinear dynamical systems and Statistical Mechanics is very fruitful for understanding principal features. It gives a new and unexpected perspective on the field of nonequilibrium and coherent pattern formation.

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