Bifurcations in Globally Coupled Shift Maps

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Abstract

A map lattice of globally coupled shift maps is investigated by analytical methods. The breakdown of the well known space time mixing regime at weak coupling strength is accompanied by a global synchronization of the maps. Furthermore a Cantor set like repeller survives which produces chaotic transients.

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Recently the problem of the influence of chaotic dynamics on spatially extended systems has become a field of intense research. In contrast to the huge number of results available from numerical studies and several phenomenological approaches there is only limited knowledge on a more rigorous level. Because the basic physical equations of motion are in general to difficult to handle with simple model systems have become popular to study the pattern formation out of a random state. One class of model systems which has been introduced basically for numerical purposes is given by coupled low dimensional maps [1, 2]. They show from a phenomenological point of view a lot of features which can be found in realistic systems also. But less is known about their solutions beyond numerical simulations. The rigorous results available so far deal mainly with hyperbolic coupled maps. It has been shown that the space time mixing regime, that means the decay of space time correlations is not affected by a sufficiently weak coupling [3, 4]. In a certain sense the coupled system behaves like the uncoupled one in this regime.

For a very simple model system I want to sketch in this note what happens if the coupling strength is increased further. For that purpose a simple model of globally coupled shift maps is considered

$$x_{n+1}^{(\nu)} = (T(\underline{x}))^{(\nu)} = 2x_n^{(\nu)} + \frac{\epsilon}{N} \sum_{\mu=0}^{N-1} \sin\left(2x_n^{(\mu)} - 2x_n^{(\nu)}\right) \quad |\bmod 2\pi \quad . \tag{1}$$

This system acts on the phase space $[0, 2\pi]^N$, ϵ denotes the coupling strength, N the system size, and the modulo operation can be ignored if the phase space is extended periodically. The coupling function $\sin(x)$ ensures that the system (1) is continuous. The following considerations are restricted to the case $N = 2$. But they can be carried over to the case of arbitrary system size [5]. The shift map admits a natural Markov partition consisting of two intervals $[0, \pi]$ and $[\pi, 2\pi]$. Its dynamics is equivalent to a shift operation of binary symbol sequences $\sigma_0, \sigma_1, \ldots$, $\sigma \in \{0, 1\}$. By a direct product this symbolic dynamics carries over to the uncoupled map lattice, $\epsilon = 0$. The Markov partition is given by hypercubes with the boundaries $x^{(\nu)} = \pi k$, $k \in \mathbb{Z}$. Every set is labeled by a symbol sequence $\sigma^{(0)} \ldots \sigma^{(N-1)}$ and the dynamics is equivalent to a shift in the two dimensional spin lattice $\left(\sigma_n^{(\nu)}\right)$. These properties can be extended to the case of finite coupling. For that purpose one considers the partition of the phase space according to the sets $(T(\underline{x}))^{(\nu)} = 2\pi k$. The implicit function theorem guarantees that these sets are well defined codimension 1 manifolds for $0 \leq \epsilon < 1$ which slice the phase space in 2^N deformed hypercubes (cf. Fig.1). The partition constructed in <Fig.1 this way coincides for $\epsilon = 0$ with the Markov partition described above. For finite coupling the preimages of this partition lead to finer partitions. In order that this symbolic dynamics is again given by the corresponding spin lattice one has to impose some expansion property $\|DT(\underline{x})\underline{y}\| \geq c\|\underline{y}\|$, $c > 1$. One can check by matrix algebra that this condition is fulfilled for $0 \leq \epsilon < 1/2$. Hence in this weak coupling regime one has obtained the result described above.

Introducing the coordinates $x^{(\pm)} = x^{(0)} \pm x^{(1)}$ the original system (1) can be rewritten as

$$x_{n+1}^{(+)} = 2x_n^{(+)} \tag{2}$$

$$x_{n+1}^{(-)} = 2x_n^{(-)} - \epsilon \sin\left(2x_n^{(-)}\right) \quad . \tag{3}$$

The first equation describes the chaotic motion along the diagonal of the phase space whereas the second equation governs the motion along the perpendicular direction. The associated map is sketched in Fig.2. For $\epsilon < 1/2$ it yields a chaotic attractor <Fig.2 which corresponds to the behaviour mentioned above. At $\epsilon = 1/2$ the map undergoes a pitchfork bifurcation. For $1/2 < \epsilon < 1$ it yields a stable fixed point $x^{(-)} = 0$ which corresponds to a synchronized solution $x_n^{(\nu)} = \xi_n$ and a chaotic Cantor set repeller. Hence the dynamics settles on a synchronized solution after a chaotic transient. A more detailed analysis of the general case shows that the transient time grows exponentially with the system size N.

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References

[1] K.Kaneko, Physica D **34**, 1 (1989).

[2] K.Kaneko, Physica D **55**, 368 (1992).

[3] L.A.Bunimovich and Ya.G.Sinai, Nonlin. **1**, 491 (1988).

[4] V.M.Gundlach and D.A.Rand, Nonlin. **6**, 165 (1993).

[5] W. Just, Globally coupled maps: Almost analytical treatment of phase transitions, submitted to Physica D, 1994.

Figure captions

Fig.1 Diagrammatic view of the Markov partition $U_{\sigma(0_{\sigma(1)}}$ for two coupled maps ($L = 2$). The phase space is 2π periodic extended to the whole plane. The dotted lined indicates the partition for the uncoupled case $\epsilon = 0$, and the broken line the partition for finite coupling $\epsilon > 0$.

Fig.2 Diagrammatic view of the map $f^{(-)}(x^{(-)}) = 2x^{(-)} - \epsilon \sin(2x^{(-)}) \, |\mathrm{mod}\, 2\pi$ for $1/2 < \epsilon < 1$. The box indicates the region which contains the repelling Cantor set.