Stationary and dynamic solutions of stochastic relaxation oscillators*

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Abstract

Two different approaches are proposed to obtain explicit solutions for stochastic relaxation oscillator problems in the weak noise limit. The first method generalizes the idea of the cumulant expansion. It does not presuppose an analytical treatment of the deterministic motion. It is however restricted to the discussion of stationary situations. In the second method an adiabatic elimination of irrelevant variables allows for the computation of time dependent solutions. It can be carried through only if the deterministic limit cycle is known analytically. As special examples the stationary solutions of the stochastic van der Pol oscillator and time dependent solutions of a simple one dimensional model system have been obtained.

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1 Introduction

The influence of small Gaussian white noise on the complicated dynamical behaviour of nonlinear dynamical systems is an area of intensive research. In addition to rather systematic investigations of simple local bifurcations [1] various techniques have been developed to treat other nonlinear problems (for a review see [2]). In this paper I am concerned with the weak noise limit of stochastic systems whose deterministic part possesses a limit cycle. Special attention is focussed on situations in which the deterministic motion is a relaxation oscillation like the famous example of the van der Pol oscillator.

For a perturbational treatment of such systems several approximation schemes can be ruled out by the following heuristic argument. In the limit of weak random perturbations the distribution function is driven by the deterministic forces towards the limit cycle. Transversally to the limit cycle the distribution remains localized while in the tangential direction small stochastic forces produce a nearly free diffusion process usually termed phase diffusion. This behaviour cannot be described by approximations which are based on the localized character of the distribution function like simple moment expansions [3]. Other approximative methods that are flexible enough to deal with these problems [4] can be applied only if the deterministic motion can be analysed analytically. Finally the discussion of the eigenvalue problem of the Fokker– Planck operator by matrix continued fractions [5] demands considerable numerical effort in the limit of weak noise.

To cope with the above mentioned situation I propose in this paper two different approximation schemes. The first one is worked out in chapter 2. It generalizes the previously developed concept of the Gaussian approximation scheme [6] and uses a physically reasonable one dimensional distrbution function as an ansatz. The resulting equations of motion simplify considerably in the limit of small diffusion and allow for the computation of stationary distribution functions. As an example the stochastic van der Pol Oscillator will be discussed in a region where it performs relaxation oscillations. A different method which allows also for the computation of time dependent solutions is presented in the third chapter. After deriving a one dimensional

Fokker– Planck equation from the full probabilistic description by adiabatic elimination of the variables transversal to the limit cycle a simple model that describes relaxation oscillations will be analysed. Although this method has the advantage to allow for the computation of time dependent solutions it requires the analytical knowledge of the deterministic limit cycle. Finally the results are summarized in a conclusion.

2 Stationary solution of a general Fokker– Planck equation

To investigate the influence of stochastic forces on sytems which possess a limit cycle I will restrict the discussion for clarity to two dimensional systems. A generalization to higher dimensions is obvious. The dynamics of the stochastic system can be described by a two dimensional Fokker– Planck equation for the distribution function.

$$\frac{\partial\rho}{\partial t} = \left(-\frac{\partial}{\partial r}D_r - \frac{\partial}{\partial\phi}D_\phi + \frac{\partial^2}{\partial r^2}D_{rr} + 2\frac{\partial^2}{\partial r\partial\phi}D_{r\phi} + \frac{\partial^2}{\partial\phi^2}D_{\phi\phi}\right)\rho(r,\phi,t) \quad (1)$$

Here r and ϕ denote the polar coordinates of the phase space and ρ the distribution function which is normalized according to $\int_0^\infty\int_0^{2\pi}\rho(r,\phi,t)\,drd\phi = 1$. The diffusion coefficients D_{rr}, $D_{r\phi}$ and $D_{\phi\phi}$ may depend on the phase space coordinates. It is furthermore assumed that the deterministic motion $\dot{r} = D_r(r,\phi)$, $\dot{\phi} = D_\phi(r,\phi)$ possesses a limit cycle. For reasons that will become clear this limit cycle must possess a parameter free representation $r = f(\phi)$. A qualitative account of the solution of this equation can be given easily in the weak noise limit. An initially localized distribution function is driven in very short times towards the limit cycle by the deterministic forces. Diffusion on the limit cycle sets in on a time scale that is determined by the inverse of the square root of the diffusion constant. Therefore the distribution function is localized in the radial direction but is widely spread out in the azimutal direction (cf. Fig.1). This simple picture will lead us in the next paragraph to an ansatz for the distribution function that allows for an easy treatment of the dynamics. The resulting equations of motion are applied in the second paragraph to the example of the van der Pol oscillator.

2.1 Modified cumulant expansion

Having in mind that the distribution function remains localized with respect to the radial variable in the course of the dynamics it seems reasonable to approximate it in this direction by a simple functional form. One has to remember that the diffusion coefficients diverge in general at $r = 0$ (cf. (16_3)). This singularity is compensated in the full Fokker– Planck equation (1) by the distribution function which vanishes at $r = 0$. It is obvious that every approximation has to fulfil this boundary condition so that a simple cumulant expansion in the radial variable which would lead to a Gaussian is excluded. Nevertheless this technical difficulty can be circumvented if one pushes the singularity via the transformation $x = \ln r$ to $-\infty$. Now in the new variables x, ϕ a cumulant expansion in the first variable up to second order is possible. It approximates the full distribution function in the x direction by a Gaussian, whose center, width and magnitude is obviously angle dependent. Going back to the old variables r, ϕ this leads to (appendix A)

$$\rho(r,\phi,t) \to \mathcal{R}(r,\phi,t) = d(\phi,t)\frac{c(\phi,t)}{r\sqrt{\pi}}\exp(-c^2(\phi,t)(\ln r - b(\phi,t))^2) \quad . \quad (2)$$

Eq.(2) can be used as an ansatz for the description of the motion which is independent of the above reasoning. It has the desired one dimensional form

→ Fig.1 (Fig.1). It is concentrated around the line $r = \exp(b(\phi))$ whereas $c^{-1}(\phi)$ denote the width and $d(\phi)$ the heigth of the distribution function. In order that this ansatz is able to describe the stochastic dynamics on the deterministic limit cycle the latter must possess a unique representation in the form $r = f(\phi)$. Otherwise eq.(2) cannot describe a distribution valid on the whole limit cycle [1]. Note that the distribution (2) is simple enough that all

[1] In those cases several parts of the limit cycle have to be treated separately and the distribution functions (2) have to be matched by continuity conditions.

its moments $\langle r^n \rangle$ can easily be computed with the help of the substitution $x = \ln r$.

The free parameters b, c, d of this distribution are fixed by suitable expectation values whose time dependence can be obtained then approximately from the Fokker– Planck equation (1). In view of the one dimensional structure of the distribution function it seems to be reasonable to describe the time evolution by moments with respect to the radial variable.

$$R^{(n)}(\phi,t)P(\phi,t) := \int r^n \rho(r,\phi,t)dr \tag{3}$$

Here

$$P(\phi,t) = \int \rho(r,\phi,t)dr \tag{4}$$

denotes the reduced probability distribution. For distribution functions that are concentrated along a line (e.g. eq.(2)) $r = R^{(1)}(\phi,t)$ represents this line and $\Delta R^{(2)} := R^{(2)} - (R^{(1)})^2$ the width of the distribution around this line. One can easily express these expectation values in terms of the parameters of the distribution (2).

$$\begin{aligned} P(\phi,t) &= d(\phi,t) \\ R^{(1)}(\phi,t) &= \exp(b(\phi,t) + \frac{1}{4c^2(\phi,t)}) \\ \Delta R^{(2)}(\phi,t) &= (R^{(1)}(\phi,t))^2 \left(\exp(\frac{1}{2c^2(\phi,t)}) - 1\right) \end{aligned} \tag{5}$$

These relations correspond to the geometrical meaning of the parameters given above.

In view of the relations (5) one may now derive a coupled system of equations for the expectation values from the equation of motion (1). In a first step one obtains immeadately by integration over r

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial \phi}\langle D_\phi \rangle_r P + \frac{\partial^2}{\partial \phi^2}\langle D_{\phi\phi} \rangle_r P \quad . \tag{6}$$

Here the right hand side has been evaluated with the help of eq.(2) and the abbreviation

$$\langle g(r,\phi) \rangle_r := \int g(r,\phi)\frac{c(\phi,t)}{r\sqrt{\pi}}\exp(-c^2(\phi,t)(\ln r - b(\phi,t))^2)dr \tag{7}$$

has been introduced. This equation represents the reduced Fokker– Planck equation in which the radial variable is averaged with the help of the ansatz (2). As a consequence the expectation value of this variable and its fluctuations (cf. eq.(5)) couple to the evolution of the reduced distribution function. Two additional equations for the radial variables can be obtained in a similar manner. Multiplication of eq.(1) with r or $(r-R^{(1)})^2$ and integration over r leads to

$$\begin{aligned}
\frac{\partial P R^{(1)}}{\partial t} &= \langle D_r\rangle_r P - \frac{\partial}{\partial\phi}\langle r D_\phi\rangle_r P - 2\frac{\partial}{\partial\phi}\langle D_{r\phi}\rangle_r P + \frac{\partial^2}{\partial\phi^2}\langle r D_{\phi\phi}\rangle_r P \\
\frac{\partial P\Delta R^{(2)}}{\partial t} &= 2\langle (r-R^{(1)})D_r\rangle_r P - \langle (r-R^{(1)})^2\frac{\partial}{\partial\phi}D_\phi\rangle_r P + 2\langle D_{rr}\rangle_r P \\
&\quad -4\langle (r-R^{(1)})\frac{\partial}{\partial\phi}D_{r\phi}\rangle_r P + \langle (r-R^{(1)})^2\frac{\partial^2}{\partial\phi^2}D_{\phi\phi}\rangle_r P \qquad (8)
\end{aligned}$$

where the right hand sides have been evaluated again with the help of the ansatz (2) [2]. Considering eq.(6) a little algebra yields

$$\begin{aligned}
P\frac{\partial R^{(1)}}{\partial t} &= \langle D_r\rangle_r P - \frac{\partial R^{(1)}}{\partial\phi}\langle D_\phi\rangle_r P - \frac{\partial}{\partial\phi}\langle (r-R^{(1)})D_\phi\rangle_r P \\
&- 2\frac{\partial}{\partial\phi}\langle D_{r\phi}\rangle_r P + 2\frac{\partial R^{(1)}}{\partial\phi}\frac{\partial}{\partial\phi}\langle D_{\phi\phi}\rangle_r P + \frac{\partial^2 R^{(1)}}{\partial\phi^2}\langle D_{\phi\phi}\rangle_r P \\
&+ \frac{\partial^2}{\partial\phi^2}\langle (r-R^{(1)})D_{\phi\phi}\rangle_r P \\
P\frac{\partial\Delta R^{(2)}}{\partial t} &= 2\langle (r-R^{(1)})D_r\rangle_r P - \frac{\partial R^{(1)}}{\partial\phi}2\langle (r-R^{(1)})D_\phi\rangle_r P \\
&- \frac{\partial}{\partial\phi}\langle (r-R^{(1)})^2 D_\phi\rangle_r P + \Delta R^{(2)}\frac{\partial}{\partial\phi}\langle D_\phi\rangle_r P + 2\langle D_{rr}\rangle_r P \\
&- 4\frac{\partial R^{(1)}}{\partial\phi}\langle D_{r\phi}\rangle_r P + 2(\frac{\partial R^{(1)}}{\partial\phi})^2\langle D_{\phi\phi}\rangle_r P - 4\frac{\partial}{\partial\phi}\langle (r-R^{(1)})D_{r\phi}\rangle_r P \\
&+ \frac{\partial^2}{\partial\phi^2}\langle (r-R^{(1)})^2 D_{\phi\phi}\rangle_r P + 2\frac{\partial^2 R^{(1)}}{\partial\phi^2}\langle (r-R^{(1)})D_{\phi\phi}\rangle_r P
\end{aligned}$$

[2] The differentation symbol $\frac{\partial}{\partial\phi}\ldots$ acts on *all* following factors, even on the distribution function that is used in calculating the expectation value $\langle\ldots\rangle_r$.

$$+ \quad 4\frac{\partial R^{(1)}}{\partial \phi}\frac{\partial}{\partial \phi}\langle (r-R^{(1)})D_{\phi\phi}\rangle_r P - \Delta R^{(2)}\frac{\partial^2}{\partial \phi^2}\langle D_{\phi\phi}\rangle_r P \quad . \tag{9}$$

Eqs.(6) and (9) constitute a closed set of coupled dynamical equations for the reduced distribution function, the expectation value of the radial variable and its fluctuations. At first sight this set seems to be as complicated as the full Fokker– Planck equation. Its structure allows however considerable simplification in the limit of small diffusion.

If the radial fluctuations are sufficiently small the following expansion is valid (appendix B)

$$\langle g(r,\phi)\rangle_r = g(R^{(1)},\phi) + \frac{\Delta R^{(2)}}{2}\frac{\partial^2 g(R^{(1)},\phi)}{\partial (R^{(1)})^2} + O((\Delta R^{(2)})^2) \quad . \tag{10}$$

In view of this relation the last term in eq.(6) the $3^{\mathrm{rd.}} - 7^{\mathrm{th.}}$ term in eq.(9_1) and the $8^{\mathrm{th.}} - 12^{\mathrm{th.}}$ term in eq.(9_2) are of the order $O(D, \Delta R^{(2)})$ and $O(D\Delta R^{(2)}, (\Delta R^{(2)})^2)$ respectively and will be neglected in the following. For the remaining terms eqs.(6) and (9) read

$$\begin{aligned}
\frac{\partial P}{\partial t} &= -\frac{\partial}{\partial \phi} D_\phi(R^{(1)},\phi)\, P + O(D,\Delta R^{(2)}) \\
\frac{\partial R^{(1)}}{\partial t} &= D_r(R^{(1)},\phi) - D_\phi(R^{(1)},\phi)\frac{\partial R^{(1)}}{\partial \phi} + O(D,\Delta R^{(2)}) \\
\frac{\partial \Delta R^{(2)}}{\partial t} &= 2\Delta R^{(2)}\left(\frac{\partial D_r(R^{(1)},\phi)}{\partial R^{(1)}} - \frac{\partial R^{(1)}}{\partial \phi}\frac{\partial D_\phi(R^{(1)},\phi)}{\partial R^{(1)}}\right) \\
&\quad -D_\phi(R^{(1)},\phi)\frac{\partial \Delta R^{(2)}}{\partial \phi} + 2D_{rr}(R^{(1)},\phi) - 4\frac{\partial R^{(1)}}{\partial \phi}D_{r\phi}(R^{(1)},\phi) \\
&\quad +2(\frac{\partial R^{(1)}}{\partial \phi})^2 D_{\phi\phi}(R^{(1)},\phi) + O(D\Delta R^{(2)}, (\Delta R^{(2)})^2) \quad .
\end{aligned} \tag{11}$$

In this approximation the equations decouple. Their structure allows for some statements about the dynamics without the computation of explicit solutions. The characteristic system belonging to eq.(11_2) coincides with the deterministic equations of motion. The central line of the distribution function follows therefore the deterministic motion and approaches the limit cycle on a time scale of order $O(1)$. This result is evidently not surprising since in the approximation of eq.(11_2) the fluctuating and diffusive contributions have been neglected. The radial fluctuations are governed by eq.(11_3) which contains a part independent of the diffusion constants that can be obtained through linearization of the "deterministic" equation (11_2). The stochastic force produces an additive term. For linearly stable deterministic systems the fluctuations scale therefore with the diffusion constant. Having this in mind the proposed approximation represents a systematic expansion with respect of the diffusion constant [3]. One may integrate the quasi linear equations (11_2) and (11_3) by the method of characteristics to obtain time dependent solutions. Further simplificatons can be made if one is interested only in stationary solutions. I will concentrate on this case in the rest of this chapter.

Eq.(11_1) yields

$$P(\phi) = Z^{-1} D_\phi^{-1}(R^{(1)}(\phi), \phi) \quad . \tag{12}$$

As one might expect the reduced probabiltiy distribution is determined in lowest order by the inverse phase space velocity on the limit cycle . Furthermore eq.($11_{2,3}$) reads in the stationary case

$$\begin{aligned} D_\phi(R^{(1)}, \phi)\frac{\partial R^{(1)}}{\partial \phi} &= D_r(R^{(1)}, \phi) \\ D_\phi(R^{(1)}, \phi)\frac{\partial \Delta R^{(2)}}{\partial \phi} &= 2\Delta R^{(2)} \left(\frac{\partial D_r}{\partial R^{(1)}} - \frac{D_r}{D_\phi}\frac{\partial D_\phi}{\partial R^{(1)}}\right) \\ &+ 2D_{rr} - 4\frac{D_r}{D_\phi}D_{r\phi} + 2(\frac{D_r}{D_\phi})^2 D_{\phi\phi} \quad . \end{aligned} \tag{13}$$

These ordinary differential equations can be solved numerically without considerable effort even for complicated dynamical systems.

[3] The linearized vesion (11) can be obtained directly from the Fokker– Planck equation (1) by a moment expansion in the radial variable.

2.2 The van der Pol oscillator

As a special example we will treat in this paragraph the stochastic van der Pol Oscillator. The deterministic equations of motion are given by

$$\begin{aligned} \dot{x} &= D_x(x,v) := v \\ \dot{v} &= D_v(x,v) := -x - \mu(x^2-1)v \quad . \end{aligned} \tag{14}$$

These equations possess a unique limit cycle which is passed by phase space points with extremely varying velocity in the limit of large μ [7]. These equations represent therefore the prototype of systems that exhibit so called relaxation oscillations. If one adds a stochastic Gaussian force of strength D to the second equation the diffusion tensor of the stochastic system is given by $D_{xx} \equiv D_{vx} \equiv 0$, $D_{vv} = D$. To put the Fokker– Planck equation in the desired form (1) one introduces polar coordinates in such a way that the limit cycle possesses a parameter free representation [8].

$$x = r\cos\phi \qquad v/\mu + x^3/3 - x = r\sin\phi \tag{15}$$

The drift and diffusion coefficients then read [4]

$$\begin{aligned} D_r &= \mu r\cos\phi(\sin\phi + \cos\phi - r^2/3\,\cos^3\phi) - r/\mu\,\sin\phi\cos\phi + D/\mu \\ D_\phi &= -\mu\sin\phi(\sin\phi + \cos\phi - r^2/3\,\cos^3\phi) - \cos^2\phi/\mu - 2D/\mu\,\sin\phi\cos\phi/r^2 \\ D_{rr} &= D/\mu\,\sin^2\phi \quad D_{r\phi} = D/\mu\,\sin\phi\cos\phi/r \quad D_{\phi\phi} = D/\mu\,\cos^2\phi/r^2 \quad . \end{aligned} \tag{16}$$

It is now an easy task to solve the boundary value problem (13). For numerical purposes it is advantageous to integrate the corresponding characteristic equations:

$$\frac{dR^{(1)}}{ds} = D_r(R^{(1)},\phi) \quad \frac{d\phi}{ds} = D_\phi(R^{(1)},\phi) \quad \frac{d\Delta R^{(2)}}{ds} = 2\Delta R^{(2)}\ldots \tag{17}$$

[4]The $O(D)$ contributions to the drift are neglected in the following as they give contributions of higher order.

Every limit cycle of this system, that is every solution with $\phi(s_1) = \phi(s_o) + 2\pi \Rightarrow R^{(1)}(s_1) = R^{(1)}(s_0)$ and $\Delta R^{(2)}(s_1) = \Delta R^{(2)}(s_0)$ yields a solution of eq.(13) in parameter representation. With the additional substitution

$$\xi := R^{(1)} \cos\phi \qquad \eta := R^{(1)} \sin\phi \tag{18}$$

eqs.(17) read explicitly

$$\begin{aligned} \frac{d\xi}{ds} &= \mu(\eta + \xi - \xi^3/3) \\ \frac{d\eta}{ds} &= -\xi/\mu \\ \frac{d\Delta R^{(2)}}{ds} &= \frac{2}{\xi^2+\eta^2}\Big(\Delta R^{(2)}(\mu\xi(\eta+\xi-\xi^3) - \xi\eta/\mu \; - 2/3\,\mu\xi^3\eta g(\xi,\eta)) \\ &\qquad + D/\mu\,(\eta - \xi g(\xi,\eta))^2\Big) \quad , \\ &\text{where} \quad g(\xi,\eta) = \frac{\xi\eta - \mu^2\xi(\xi+\eta-\xi^3/3)}{\xi^2+\mu^2\eta(\xi+\eta-\xi^3/3)} \quad . \end{aligned} \tag{19}$$

→ Fig.2 The results of the numerical integration for the parameter values $\mu = 5$ and $\mu = 1$ are shown in Fig.2. Eq.($19_{1,2}$) yields the deterministic limit cycle which is shown in the first picture and forms the center line of the distribution function. The reduced distribution as a function of the two "cartesian" coordinates is shown in the second and third pictures respectively. It can be computed easily with the help of eq.(12). The fact that the phase space points of the deterministic system pass the horizontal (vertical) branches of the limit cycle with high (low) velocity in the limit of large μ values is obviously reflected by this distribution because it is determined by the inverse of the phase space velocity. Finally the fourth and fifth pictures show the dependence of the radial fluctuations. Especially in the case $\mu = 5$ one obtains a dramatic enhancement of the fluctuations (about 3 orders of magnitude) along the horizontal branches. Referring back to the deterministic system this behaviour can be explained qualitatively by the following two arguments. On one hand the deterministic forces vary strongly if the system passes from the veritcal to the horizontal branches. It is well known that this behaviour leads to an enhancement of the fluctuations which is similar to the fluctuation enhancement in the vicinity of an unstable equilibrium point [9]. On the other hand the deterministic forces which act in the horizontal direction suppress the radial fluctuations only along the vertical branches. This mechanism is not present along the horizontal branches of the limit cycle so that the stochastic force is able to increase the radial fluctuations.

In concluding this section I want to point out that the method explained above allows for the computation of stationary solutions of arbitrary two dimensional linearly stable stochastic systems in the limit of weak noise. An analytical knowledge of the limit cycle is not necessary so that one is not forced to put the problem into appropriate coordinates. On the other hand the method cannot be applied easily in the vicinity of bifurcation points and for the computation of time dependent solutions. A strategy for the investigation of the latter problem will be developed in the next section.

3 Time dependent solutions of reduced Fokker– Planck equations

Regarding the discussion at the beginning of chapter 2 there remains another strategy for the solution of the Fokker– Planck equation in the weak noise limit. On a time scale that is given by the inverse square root of the diffusion constant the distribution function is concentrated around the deterministic limit cycle. In this domain it seems possible that the radial variable can be eliminated from the description.

We will again treat the system described by eq.(1). In this chapter the smallness of the diffusion constants is made explicit by an expansion parameter ϵ^2 ($D_{rr} \to \epsilon^2 D_{rr}$, etc. in eq.(1)). One has now to assume that the stable limit cycle of the deterministic system is known analytically. Without loss of generality it can be put into the form $r = 1$ at least locally [5]. This means that $D_r(1, \phi) = 0$ and $D_\phi(1, \phi) > 0$ hold. A very simple elimination of the

[5]If for example the system is given in cartesian coordinates and $x = \tau_1(\psi)$, $y = \tau_2(\psi)$, $\psi \in [0, 2\pi)$ denotes a parameter representation of the limit cycle then the explicit expressions for the drift and diffusion coefficients of the Fokker– Planck equation (1) can be obtained with the help of the transformation $x = r\tau_1(\phi)$, $y = r\tau_2(\phi)$.

radial variable consists in a complete neglection of its fluctuations. Approximating the radial dependence of the drift and diffusion coefficients by their values on the deterministic limit cycle one obtains the "bare" Fokker– Planck equation for the reduced distribution function.

$$\frac{\partial P}{\partial t} = \left(-\frac{\partial}{\partial \phi} D_\phi(1,\phi) + \epsilon^2 \frac{\partial^2}{\partial \phi^2} D_{\phi\phi}(1,\phi) \right) P(\phi,t) \tag{20}$$

This equation will be renormalized if one takes the fluctuating character of the radial variable into account [10]. It is the main objective of the next paragraph to show how this renormalization procedure can be carried through and under what conditions the resulting equation has again the form of a Fokker– Planck equation. Finally we will treat a simple model system which describes a relaxation oscillator in this way.

3.1 Adiabatic elimination of the radial variable

In order to eliminate the radial variable we expand the Fokker– Planck equation with the help of the transformation $r = 1 + \epsilon s$ around the deterministic limit cycle. Using the formal expansions

$$\begin{aligned}
D_r(1+\epsilon s,\phi) &= \epsilon s D_r^{(1)}(\phi) + \frac{(\epsilon s)^2}{2} D_r^{(2)}(\phi) + \frac{(\epsilon s)^3}{3!} D_r^{(3)}(\phi) + \ldots \\
D_\phi(1+\epsilon s,\phi) &= D_\phi^{(0)}(\phi) + \epsilon s D_\phi^{(1)}(\phi) + \frac{(\epsilon s)^2}{2} D_\phi^{(2)}(\phi) + \ldots \\
D_{rr}(1+\epsilon s,\phi) &= D_{rr}^{(0)}(\phi) + \epsilon s D_{rr}^{(1)}(\phi) + \frac{(\epsilon s)^2}{2} D_{rr}^{(2)}(\phi) + \ldots \\
D_{r\phi}(1+\epsilon s,\phi) &= D_{r\phi}^{(0)}(\phi) + \epsilon s D_{r\phi}^{(1)}(\phi) + \ldots \\
D_{\phi\phi}(1+\epsilon s,\phi) &= D_{\phi\phi}^{(0)}(\phi) + \ldots
\end{aligned} \tag{21}$$

eq.(1) reads [6]

$$\frac{\partial \rho}{\partial t} = -\mathbf{\Lambda}\rho = -(\mathbf{\Lambda_0} + \epsilon \mathbf{\Lambda_1} + \epsilon^2 \mathbf{\Lambda_2} + \ldots)\rho \tag{22}$$

[6] In the rest of this paragraph it will be assumed that $s \in [-\infty, \infty]$. The resulting error is of exponential order in $-\epsilon^{-1}$.

where the abbreviations

$$\begin{aligned}
\mathbf{\Lambda_0} &= \frac{\partial}{\partial s} s D_r^{(1)} + \frac{\partial}{\partial \phi} D_\phi^{(0)} - \frac{\partial^2}{\partial s^2} D_{rr}^{(0)} \\
\mathbf{\Lambda_1} &= \frac{\partial}{\partial s}\frac{s^2}{2} D_r^{(2)} + \frac{\partial}{\partial \phi} s D_\phi^{(1)} - \frac{\partial^2}{\partial s^2} s D_{rr}^{(1)} - 2\frac{\partial^2}{\partial s \partial \phi} D_{r\phi}^{(0)} \\
\mathbf{\Lambda_2} &= \frac{\partial}{\partial s}\frac{s^3}{6} D_r^{(3)} + \frac{\partial}{\partial \phi}\frac{s^2}{2} D_\phi^{(2)} - \frac{\partial^2}{\partial s^2}\frac{s^2}{2} D_{rr}^{(2)} \\
&\quad -2\frac{\partial^2}{\partial s \partial \phi} s D_{r\phi}^{(1)} - \frac{\partial^2}{\partial \phi^2} D_{\phi\phi}^{(0)}
\end{aligned} \tag{23}$$

have been introduced. For a perturbational treatment of this equation an explicit solution of the eigenvalue problem of $\mathbf{\Lambda_0}$ is necessary. The corresponding stochastic differential equations $\dot{s} = -sD_r^{(1)}(\phi) + \epsilon\sqrt{2D_{rr}^{(0)}}\,\xi(t)$, $\dot{\phi} = D_\phi^{(0)}(\phi)$ describe a deterministic rotator which drives parametrically an Ornstein– Uhlenbeck process. As these equations can be integrated directly it is obvious that the eigenvalue problem of $\mathbf{\Lambda_0}$ possesses an elementary solution. The eigenfunctions are products from those of the rotator and from the Ornstein– Uhlenbeck process. The solution turns out to be

$$\begin{aligned}
\mathbf{\Lambda_0} u_{nm} &= \lambda_{nm} u_{nm} \\
\lambda_{nm} &= \frac{A(2\pi)}{G(2\pi)} n - \frac{2\pi i}{G(2\pi)} m \\
u_{nm}(s,\phi) &= Z_n \frac{\exp(-nB(\phi))}{D_\phi^{(0)}} \exp(-2\pi i m \frac{G(\phi)}{G(2\pi)})\, c(\phi) H_n(cs) \exp(-c^2 s^2) \quad .
\end{aligned} \tag{24}$$

with

$$G(\phi) = \int_0^\phi \frac{d\phi'}{D_\phi^{(0)}(\phi')}, \quad A(\phi) = \int_0^\phi \frac{D_r^{(1)}(\phi')}{D_\phi^{(0)}(\phi')} d\phi' \quad . \tag{25}$$

The $H_n(x)$ denote the Hermite polynominals. Expressions for the remaining terms $B(\phi)$, $c(\phi)$ and Z_n can be found in appendix C. Note that $G(2\pi) > 0$ holds by assumption whereas $A(2\pi) > 0$ is valid due to the stability of the deterministic limit cycle. Therefore we have $\mathrm{Re}\,\lambda_{nm} \geq 0$ so that $\mathbf{\Lambda_0}$

is bounded from below and is a reasonable Fokker– Planck operator. If $(a|b) := \int\int a^* b \, ds d\phi$ denotes the usual scalar product and $\mathbf{\Lambda_0^\dagger}$ the hermitian conjugate operator to $\mathbf{\Lambda_0}$ one gets

$$\begin{aligned}\mathbf{\Lambda_0^\dagger} v_{nm} &= \lambda^*_{nm} v_{nm} \\ v_{nm}(s,\phi) &= Z_n \exp(nB(\phi)) \exp(-2\pi i m \frac{G(\phi)}{G(2\pi)}) H_n(cs) \quad . \end{aligned} \tag{26}$$

The functions $\{u_{nm}, v_{nm}\}$ represent a complete biorthogonal system [5]

$$(v_{n'm'}|u_{nm}) = \delta_{n'n}\delta_{m'm} \quad \mathbf{1} = \sum_{nm} |u_{nm})(v_{nm}| \quad . \tag{27}$$

For the elimination of the radial variable s the projection operator method [11] is chosen. With the help of the projection operator onto the subspace of $\mathbf{\Lambda_0}$ with eigenvalues $\mathrm{Re}\,\lambda_{nm} = 0$

$$\mathbf{P}\ldots := \sum_m |u_{0m})(v_{0m}|\ldots) = \frac{c(\phi)}{\sqrt{\pi}} \exp(-c^2 s^2) \int \ldots ds' \tag{28}$$

one may define a relevant distribution function

$$\rho_{rel}(s,\phi,t) := \mathbf{P}\rho(s,\phi,t) = \frac{c(\phi)}{\sqrt{\pi}} \exp(-c^2 s^2) P(\phi,t) \quad . \tag{29}$$

Its ϕ– dependent part mirrors the reduced distribution function. The radial variable is Gaussian distributed around the limit cycle $s = 0$. If the initial distribution has the form of a relevant distribution function the time evolution can be described by the integro differential equation

$$\frac{\partial \rho_{rel}}{\partial t} = -\mathbf{P\Lambda P}\rho_{rel} + \int_0^t \mathbf{P\Lambda Q} e^{-\mathbf{Q\Lambda Q}t'} \mathbf{Q\Lambda P} \rho_{rel}(t-t')dt' \quad . \tag{30}$$

Here $\mathbf{Q} = \mathbf{1} - \mathbf{P}$ denotes the complement of the projector $\mathbf{P}$. As by eq.(24_1) (26_1) and (28) $\mathbf{\Lambda_0}$ commutes with $\mathbf{P}$, $\mathbf{P\Lambda Q}$ and $\mathbf{Q\Lambda P}$ are of first order in ϵ. Usual pertubation theory up to second order yields

$$\begin{aligned}\frac{\partial \rho_{rel}}{\partial t} &= -\mathbf{P}(\mathbf{\Lambda_0} + \epsilon\mathbf{\Lambda_1} + \epsilon^2\mathbf{\Lambda_2})\mathbf{P}\rho_{rel}(t) \\ &\quad + \epsilon^2 \int_0^t \mathbf{P\Lambda_1 Q} e^{-\mathbf{\Lambda_0}t'} \mathbf{Q\Lambda_1 P}\rho_{rel}(t-t')dt' + O(\epsilon^3) \quad . \end{aligned} \tag{31}$$

The further evaluation is a standard problem in statistical mechanics because $\mathbf{\Lambda_0}$ has been diagonalized (appendix C). One obtains

$$\begin{aligned}\frac{\partial P}{\partial t} &= \left(-\frac{\partial}{\partial\phi}(D_\phi^{(0)} + \epsilon^2\frac{D_\phi^{(2)}}{4c^2}) + \epsilon^2\frac{\partial^2}{\partial\phi^2}D_{\phi\phi}^{(0)}\right) P(\phi,t) \\ &+ \epsilon^2\frac{\partial}{\partial\phi}\frac{D_\phi^{(1)}}{c(\phi)}\int_0^t\int_0^{2\pi} K(t',\phi,\phi')\exp(B(\phi')-B(\phi))\left(-\frac{D_r^{(2)}(\phi')}{4c(\phi')}\right. \\ &+ \left. 2c(\phi')\frac{\partial}{\partial\phi'}(D_{r\phi}^{(0)}(\phi') + \frac{D_\phi^{(1)}(\phi')}{4c^2(\phi')})\right) P(\phi',t-t')d\phi' dt' \quad . \qquad (32)\end{aligned}$$

The kernel is given by

$$K(t,\phi,\phi') = \exp(-\frac{A(2\pi)}{G(2\pi)}t)\frac{2\pi}{G(2\pi)D_\phi^{(0)}(\phi)}\delta_{2\pi}(\frac{2\pi}{G(2\pi)}(G(\phi)-G(\phi')-t)) \quad (33)$$

where $\delta_{2\pi}(x)$ denotes the 2π periodic continuation of the $\delta-$ function. Eq.(32) represents the renormalization of eq.(20) by the radial fluctuations. The drift coefficient is corrected by the equilibrium fluctuations as can be seen by inspecting eq.(21_2) ($2^{\text{nd.}}$ term) and the equilibrium distribution (cf. eq.(29)). The diffusion coefficient experiences a nonlocal correction. Its time dependence is determined by the kernel (33) which produces a retardation in the angle variable via the undisturbed motion. The exponential decay of the retardation is given by the smallest nonvanishing real part of the eigenvalues (24_2). Also this contribution to the Fokker– Planck equation comes from the fluctuations of the radial variable. But in contrast to the term described above this term emerges as a consequence of the deterministic motion that influences also the steady state solution due to a non vanishing probability current. In this sense this term describes the dynamic fluctuations of the system. If the relaxation of the radial variable towards the limit cycle is fast enough, i.e. $A(2\pi) \gg 1$, the dynamic fluctuations act instantaneously on the angle variable. In this case the exponential in eq.(33) decays so fast that it can be approximated by a $\delta-$ function (Markovian approximation)

$$A(2\pi) \gg 1 : \qquad \exp(-\frac{A(2\pi)}{G(2\pi)}t) \to \frac{G(2\pi)}{A(2\pi)}\delta(t-0) \quad . \qquad (34)$$

Eq.(32) then reads

$$\frac{\partial P}{\partial t} = \left(-\frac{\partial}{\partial \phi}(D_\phi^{(0)} + \epsilon^2 \frac{D_\phi^{(2)}}{4c^2}) + \epsilon^2 \frac{\partial^2}{\partial \phi^2} D_{\phi\phi}^{(0)} \right) P(\phi, t)$$
$$- \quad \epsilon^2 \frac{G(2\pi)}{A(2\pi)} \frac{\partial}{\partial \phi} \left(\frac{D_\phi^{(1)} D_r^{(2)}}{4c^2} + \frac{\partial D_\phi^{(1)}}{\partial \phi}(D_{r\phi}^{(0)} + \frac{D_\phi^{(1)}}{4c^2}) \right) P(\phi, t)$$
$$+ \quad \epsilon^2 \frac{G(2\pi)}{2A(2\pi)} \frac{\partial^2}{\partial \phi^2} D_\phi^{(1)} (D_{r\phi}^{(0)} + \frac{D_\phi^{(1)}}{4c^2}) P(\phi, t) \quad . \tag{35}$$

This equation shows that the time evolution of an arbitrary stochastic system on the limit cycle can be described by an one dimensional Fokker– Planck equation in the limit of weak diffusion and the Markovian approximation. Up to corrections which stem from the radial fluctuations the drift coefficient is given by the deterministic term. The diffusion coefficient suffers a complicated looking correction. Even in the special case that the stochastic force acts only on the radial variable ($D_{\phi\phi} \equiv D_{r\phi} \equiv 0$) eq.(35) possesses a nonvanishing diffusion coefficient $\sim (D_\phi^{(1)})^2$. This is a consequence of the coupling of the stochastic radial variable to the angle variable (cf. eq.(21_2)).

3.2 Discussion of a one dimensional model system

Although the considerations of the preceeding paragraph allow an explicit calculation of the effective drift and diffusion coefficients their actual evaluation may require a considerable computational effort. As only some basic features should be discussed in this paper we will consider a very simplified model for a relaxation oscillator.

It is obvious that a Fokker– Planck equation with a constant diffusion coefficient can be obtained from the general equation (35) by a simple coordinate transformation $\phi \to \psi$. It is furthermore obvious that in all cases where the original diffusion coefficient has only a weak coordinate dependence the drift coefficient remains essentially unchanged. As we focus on systems exhibiting relaxation oscillations this means that the new drift coefficient $D_\psi =: -V'(\psi)$ describes a motion with extremely varying phase space velocity. As a consequence the "potential" $V(\psi)$ possesses regions with extremely small slope. The regions on the limit cycle with large phase space velocity that means with large slope of the potential give rise only to small quantitative contributions to the time dependent solutions and can be handled in a crude manner. By this reasoning one may conclude that the essential part of the motion may be described by the potential $V(\psi) = -A\psi + \sin\psi$ where ψ denotes the 2π periodic variable on the limit cycle. In the case $A > 1$ the region $\psi \sim 0$ is passed by the phase space points with extremely low velocity and reflects thus the behaviour of a relaxation oscillator. Having this in mind the motion of the stochastic system is governed by the following Fokker– Planck equation

$$\frac{\partial P}{\partial t} = \left(-\mathbf{\Lambda_0} - \epsilon^2\mathbf{\Lambda_1}\right) P = \left(\frac{\partial}{\partial\psi}V'(\psi) + \epsilon^2 D\frac{\partial^2}{\partial\psi^2}\right) P(\psi,t) \quad . \tag{36}$$

This equation has been studied in the literature in different contexts (e.g. [5]). The new aspect introduced in this paper is that one may regard eq.(36) for the case $A > 1$ on which I will concentrate as a kind of "normal form" for a stochastic relaxation oscillator.

The most powerful method to get time dependent solutions for Fokker– Planck equations is the analysis of the corresponding eigenvalue problem. Due to the positivity of the drift and the smallness of the diffusion coefficient a simple perturbative treatment can be applied which allows for an analytical calculation of the eigenvalues and eigenfunctions. The eigenvalue problem for the Fokker– Planck operator and its hermitian conjugate [7] reads

$$(\mathbf{\Lambda_0} + \epsilon^2\mathbf{\Lambda_1})u_n = \lambda_n u_n \qquad (\mathbf{\Lambda_0^\dagger} + \epsilon^2\mathbf{\Lambda_1^\dagger})v_n = \lambda_n^* v_n \quad . \tag{37}$$

Assuming that the functions $\{u_n, v_n\}$ constitute a complete biorthogonal set, a simple expansion of the eigenvalues and eigenfunctions

$$\begin{aligned} \lambda_n &= \lambda_n^{(0)} + \epsilon^2\lambda_n^{(1)} + \cdots \\ u_n &= u_n^{(0)} + \epsilon^2 u_n^{(1)} + \cdots \\ v_n &= v_n^{(0)} + \epsilon^2 v_n^{(1)} + \cdots \end{aligned} \tag{38}$$

[7]With respect to the canonical scalar product $(a|b) := \int_0^{2\pi} a^*(\psi)b(\psi)\,d\psi$.

yields for the time dependent solutions

$$\begin{aligned}P(t) &= \sum_n ((v_n^{(0)}|P(0)) + \epsilon^2(v_n^{(1)}|P(0)) + \cdots)\exp(-(\lambda_n^{(0)} + \epsilon^2\lambda_n^{(1)} + \cdots)t) \\ &\quad \cdot(u_n^{(0)} + \epsilon^2 u_n^{(1)} + \cdots) \\ &= P_0(t, \epsilon^2 t, \ldots) + \epsilon^2 P_1(t, \epsilon^2 t, \ldots) + \cdots \quad . \end{aligned} \tag{39}$$

If one is interested in the long time behaviour it is sufficient to restrict the expansion to the $\epsilon^2 t$ terms of the distribution function. Consequently one has to calculate the eigenvalues only up to first order in ϵ^2. Terms of higher order give rise to neglegible corrections. Therefore we will restrict ourself to the approximation

$$P(t) \simeq \sum_n (v_n^{(0)}|P(0))\exp(-(\lambda_n^{(0)} + \epsilon^2\lambda_n^{(1)})t)u_n^{(0)} \quad . \tag{40}$$

The form (39_2) which is produced by by the perturbation expansion of the eigenvalue problem corresponds to the ansatz used in the multiple time scaling method [12]. It is easy to convince oneself that the approximation (40) agrees with the corresponding truncation of the multiple time scaling hierarchy (appendix D).

Using eq.(37) and (38) one gets

$$\mathbf{\Lambda_0} u_n^{(0)} = \lambda_n^{(0)} u_n^{(0)} \qquad \mathbf{\Lambda_0^\dagger} v_n^{(0)} = \lambda_n^{(0)*} v_n^{(0)} \qquad \lambda_n^{(1)} = (v_n^{(0)}|\mathbf{\Lambda_1} u_n^{(0)}) \tag{41}$$

from which the explicit expressions

$$\begin{aligned} u_n^{(0)}(\psi) &= -G^{-1/2}(2\pi)V'^{-1}(\psi)\exp(2\pi i n\frac{G(\psi)}{G(2\pi)}) \\ v_n^{(0)}(\psi) &= G^{-1/2}(2\pi)\exp(2\pi i n\frac{G(\psi)}{G(2\pi)}) \\ \lambda_n^{(0)} &= \frac{2\pi i n}{G(2\pi)} \\ \lambda_n^{(1)} &= -\frac{D}{G^3(2\pi)}(2\pi n)^2\int_0^{2\pi}\frac{d\psi}{V'^3(\psi)} = D\frac{A^2+\frac{1}{2}}{A^2-1}n^2 \end{aligned} \tag{42}$$

may be derived. Here the abbreviation

$$G(\psi) := -\int_0^{\psi} \frac{d\psi'}{V'(\psi')} \tag{43}$$

has been introduced. Looking at eq.(42_4) it is evident that the perturbation expansion leads to useful results only in the case $A-1 \gg \epsilon^2$. If this condition is not fulfilled the terms of first and higher order in ϵ^2 are of the same order of magnitude so that the perturbation series cannot be truncated at any finite order. If one specializes to the important initial condition $P(\psi,0) = \delta(\psi-\psi_0)$ eq.(40) yields the conditional probability distribution $P(\psi,t|\psi_0)$. It reads explicitly

$$P(\psi,t|\psi_0) \simeq -G^{-1}(2\pi)\sum_n V'^{-1}(\psi)\exp\left(i2\pi n\frac{G(\psi)-G(\psi_0)-t}{G(2\pi)} - \epsilon^2\lambda_n^{(1)}t\right) \quad . \tag{44}$$

Its time dependence is governed on the one hand by a term which corresponds to the deterministic motion on the limit cycle. On the other hand the eigenfunctions are damped out according to the real part of the eigenvalues which means that the distribution function broadens and approaches the stationary solution. To get further insight into the time evolution one can compute the first Fourier coefficients $\langle\sin\psi\rangle$, $\langle\cos\psi\rangle$ of the distribution function. Specializing to the initial condition $P(\psi,0) = \delta(\psi)$ a little algebra yields

$$\begin{aligned} \langle\sin\psi\rangle &= -2\sqrt{A^2-1}\sum_{n=1}^{\infty}(\sqrt{A^2-1}-A)^n \sin\frac{2\pi n t}{G(2\pi)}\,\exp(-\epsilon^2\lambda_n^{(1)}t) \\ \langle\cos\psi\rangle &= A-\sqrt{A^2-1}+(\sqrt{A^2-1}-A) \\ &\quad \cdot 2\sqrt{A^2-1}\sum_{n=1}^{\infty}(\sqrt{A^2-1}-A)^n \cos\frac{2\pi n t}{G(2\pi)}\,\exp(-\epsilon^2\lambda_n^{(1)}t) \end{aligned} \tag{45}$$

→ Fig.3 Fig.3 shows the result for two different values of A. Initially the solutions show an oscillating behaviour whose extremly varying phase space velocity corresponds to the relaxation oscillation of the deterministic system. As time goes on the initially localized distribution function broadens which causes a damping of the expectation values. We find an exponential decay for large times which is determined by the eigenvalue $\epsilon^2\lambda_1^{(1)}$ (cf. eq.(42_4)). It is therefore more pronounced in the limit of small $A-1$ values i.e. if well developed relaxation oscillations are present. This seems to be a reasonable result. As a relaxation oscillation somehow indicates that a saddle node bifurcation occurs on the limit cycle [13] the stochastic system will show an enhancement of the fluctuations in the vicinity of the bifurcation point. This effect is similar to the enhancement of fluctuations in the presence of unstable fixed points [9] and leads to an extremely fast thermalization of the distribution function.

4 Conclusion

Two different approaches have been presented to discuss stochastic systems in the weak noise limit in the presence of a limit cycle. Special attention has been payed to the relaxation oscillation behaviour. The method developed in chapter 2 generalizes the ideas of cumulant expansion techniques. It has the advantage that one does not need to integrate the deterministic equations analytically. On the other hand it allows only for an efficient computation of stationary solutions of linearly stable systems. To analyse time dependent behaviour also it was shown in the third chapter that the stochastic dynamics on the limit cycle can be described by an effective one dimensional Fokker–Planck equation. For the case that the deterministic limit cycle is known analytically explicit expressions for the effective drift and diffusion coefficients have been given. To obtain the time dependent solutions themselves a simple perturbation expansion may be used. This program was carried out for a simple model system that reflects the chief features of relaxation oscillations. An important result is that a well developed relaxation oscillation leads to a fast thermalization of the distribution function. It is hoped that the ideas developed in this context may be used for the analysis of more complicated dynamical systems.

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Appendix A

With the help of the transformation $x = \ln r$ one introduces a new distribution function $\hat{\rho}(x, \phi, t)\, dx d\phi = \rho(r, \phi, t)\, dr d\phi$. To derive the ansatz (2) let us introduce the cumulants $K_\phi^{(n)}$ of the first variable using the characteristic function

$$\int \exp(i\alpha x)\hat{\rho}(x, \phi, t)dx = \exp(K(\alpha, \phi, t)) = \exp\left(\sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} K_\phi^{(n)}(t)\right) \quad . \tag{46}$$

The first cumulants coincide with the lowest moments of the distribution with respect to the first variable.

$$\begin{aligned} \exp(K_\phi^{(0)}) &= \int \hat{\rho}(x, \phi, t)dx \\ K_\phi^{(1)} &= \int x\hat{\rho}(x, \phi, t)dx \\ K_\phi^{(2)} &= \int (x - K_\phi^{(1)})^2 \hat{\rho}(x, \phi, t)dx \end{aligned} \tag{47}$$

If one neglects cumulants of higher order eq.(46) can be solved for the distribution function yielding a Gaussian. Going back to the old variable r one gets eq.(2) if one identifies

$$\exp(K_\phi^{(0)}) = d(\phi, t) \quad K_\phi^{(1)} = b(\phi, t) \quad K_\phi^{(2)} = c^{-2}(\phi, t)/2 \quad . \tag{48}$$

If the distribution $\hat{\rho}$ is localized in the variable $x = \ln r$ that means that the distribution ρ is localized in r the neglection of the higher order cumulants seems to be a reasonable lowest order approximation.

Let me finally stress that it is also possible to derive a closed set of equations for the cumulants. If one transforms eq.(1) to the new variable x [8]

$$\begin{aligned}\frac{\partial\hat{\rho}}{\partial t} &= \left(-\frac{\partial}{\partial x}(e^{-x}D_r - e^{-2x}D_{rr}) - \frac{\partial}{\partial\phi}D_\phi\right.\\ &\left.+\frac{\partial^2}{\partial x^2}e^{-2x}D_{rr} + 2\frac{\partial^2}{\partial x\partial\phi}e^{-x}D_{r\phi} + \frac{\partial^2}{\partial\phi^2}D_{\phi\phi}\right)\hat{\rho}(x,\phi,t) \qquad (49)\end{aligned}$$

one gets tracing back to eq.(46)

$$\begin{aligned}\frac{\partial}{\partial t}\exp(K(\alpha,\phi,t)) &= \iint\Big(i\alpha(e^{-x}D_r - e^{-2x}D_{rr})\\ &-\frac{\partial}{\partial\phi}D_\phi - \alpha^2e^{-2x}D_{rr} - 2i\alpha\frac{\partial}{\partial\phi}e^{-x}D_{r\phi} + \frac{\partial^2}{\partial\phi^2}D_{\phi\phi}\Big)\\ &\cdot 2\pi\exp(i(\alpha-\alpha'))dx\exp(K(\alpha',\phi,t))d\alpha' \qquad (50)\end{aligned}$$

Using $K_\phi^{(n)} = \frac{\partial^n K}{\partial\alpha^n}|_{\alpha=0}$ and the expansion in eq.(47) a coupled and closed set of equations for the cumulants can be obtained. If one truncates this set at cumulants of second order the resulting system differ from the eqs.(6) and (9) obtained in this paper. The main difference comes from the fact that in general no simple relation between $K_\phi^{(n)}$ and $R^{(n)}$ existst. A tedious computation shows however that both approaches coincide in the limit of weak diffusion treated in chapter 2.

Appendix B

Eq.(7) and (5) yield

$$\langle r^n\rangle_r = \exp(bn + \frac{n^2}{4c^2}) = (R^{(1)})^n\left(1 + \frac{\Delta R^{(2)}}{(R^{(1)})^2}\right)^{n(n-1)/2} \quad . \qquad (51)$$

In view of the relations

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\binom{n}{k} &= 0 \qquad (n\geq 1)\\ \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k}{2} &= 0 \qquad (n\geq 3) \qquad (52)\end{aligned}$$

[8]The coefficients have to be interpreted as functions of x and ϕ eg. $D_r = D_r(r,\phi) = D_r(e^x,\phi)$.

one gets

$$
\begin{aligned}
&\langle (r-R^{(1)})^n\rangle_r \\
=\ & \sum_{k=0}^{n}(-1)^{n-k}\left(\begin{array}{c} n \\ k \end{array}\right)\langle r^k\rangle_r (R^{(1)})^{n-k} \\
=\ & (R^{(1)})^n \sum_{k=0}^{n}(-1)^{n-k}\left(\begin{array}{c} n \\ k \end{array}\right)\left(1+\frac{k(k-1)}{2}\frac{\Delta R^{(2)}}{(R^{(1)})^2}+O((\Delta R^{(2)})^2)\right) \\
=\ & O((\Delta R^{(2)})^2) \qquad (n\geq 3) \quad . \qquad (53)
\end{aligned}
$$

With the help of a Taylor expansion one obtains finally

$$
\begin{aligned}
\langle g(r,\phi)\rangle_r &= \sum_{\nu=0}^{\infty}\frac{1}{\nu!}\frac{\partial^\nu g(R^{(1)},\phi)}{\partial (R^{(1)})^\nu}\langle (r-R^{(1)})^\nu\rangle_r \\
&= g(R^{(1)},\phi)+\frac{\partial^2 g(R^{(1)},\phi)}{\partial (R^{(1)})^2}\frac{\Delta R^{(2)}}{2}+O((\Delta R^{(2)})^2) \quad . \qquad (54)
\end{aligned}
$$

Appendix C

The explicit expressions for the normalization constant Z_n and the functions $B(\phi)$ and $c(\phi)$ read

$$
\begin{aligned}
Z_n &= (G(2\pi)2^n n!\sqrt{\pi})^{-1/2} \\
B(\phi) &= A(\phi)-\frac{A(2\pi)}{G(2\pi)}G(\phi)-\ln c(\phi) \\
\left(c^{-2}\right)' &= -2\frac{D_r^{(1)}(\phi)}{D_\phi^{(0)}(\phi)}c^{-2}+\frac{4D_{rr}^{(0)}(\phi)}{D_\phi^{(0)}(\phi)} \qquad c(\phi)=c(\phi+2\pi) \quad . \qquad (55)
\end{aligned}
$$

We do not need these expressions for the computations of the frequency and integral term.

Frequency term: With the definitions (28) and (29) one obtains taking the explicit expressions (23) into account

$$
\begin{aligned}
&\mathbf{P}(\mathbf{\Lambda_0}+\epsilon\mathbf{\Lambda_1}+\epsilon^2\mathbf{\Lambda_2})\mathbf{P}\rho_{rel.}(t-t') \\
=\ & \frac{c(\phi)}{\sqrt{\pi}}\exp(-c^2s^2)\int\left(\frac{\partial}{\partial\phi}D_\phi^{(0)}+\epsilon\frac{\partial}{\partial\phi}s'D_\phi^{(1)}+\epsilon^2\frac{\partial}{\partial\phi}\frac{s'^2}{2}D_\phi^{(2)}-\epsilon^2\frac{\partial^2}{\partial\phi^2}D_{\phi\phi}^{(0)}\right)
\end{aligned}
$$

$$\cdot \frac{c(\phi)}{\sqrt{\pi}} \exp(-c^2 s'^2) ds' P(\phi, t)$$

$$= \frac{c(\phi)}{\sqrt{\pi}} \exp(-c^2 s^2) \left(\frac{\partial}{\partial \phi} D_\phi^{(0)} + \epsilon^2 \frac{\partial}{\partial \phi} \frac{D_\phi^{(2)}}{4c^2(\phi)} - \epsilon^2 \frac{\partial^2}{\partial \phi^2} D_{\phi\phi}^{(0)} \right) P(\phi, t) \quad . \quad (56)$$

In the last step the orthogonality relations of the Hermite polynomials have been used.

Integral term: Using again the definitions (28), (29) and the orthogonality relations of the Hermite polynomials together with the explicit expression of $\mathbf{\Lambda_1}$ (cf. eq.(23_2)) one gets

$$\mathbf{P\Lambda_1 Q} e^{-\mathbf{\Lambda_0} t'} \mathbf{Q \Lambda_1 P} \rho_{rel.}(t - t')$$

$$= \sum_{m,m',m''} |u_{0m})(v_{0m}| \frac{\partial}{\partial \phi} s D_\phi^{(1)} u_{1m'}) \cdot \overbrace{(v_{1m'}| e^{-\mathbf{\Lambda_0} t'} u_{1m'})}^{=\exp(-\lambda_{1m'} t')}$$

$$\cdot \quad (v_{1m'}| \left(\frac{\partial}{\partial s} \frac{s^2}{2} D_r^{(2)} + \frac{\partial}{\partial \phi} s D_\phi^{(1)} - 2 \frac{\partial^2}{\partial s \partial \phi} D_{r\phi}^{(0)} \right) u_{0m''})(v_{0m''}| \rho_{rel.}(t - t')) \quad . \quad (57)$$

In view of eq.(28) the first factor can be put into the form

$$\sum_m |u_{0m})(v_{0m}| \frac{\partial}{\partial \phi} s D_\phi^{(1)} u_{1m'})$$

$$= \frac{c(\phi)}{\sqrt{\pi}} \exp(-c^2 s^2) \int \frac{\partial}{\partial \phi} s' D_\phi^{(1)}(\phi) u_{1m'}(s', \phi) ds'$$

$$= \frac{c(\phi)}{\sqrt{\pi}} \exp(-c^2 s^2) \frac{\partial}{\partial \phi} \frac{D_\phi^{(1)} \exp(-B(\phi))}{2 Z_1 G(2\pi) D_\phi^{(0)} c(\phi)} \exp\left(-2\pi i m \frac{G(\phi)}{G(2\pi)} \right) \quad .(58)$$

In a similar way the third factor yields with respect to eq.(28), (29) and a partial integration in s

$$\sum_{m''} (v_{1m'}| \left(\frac{\partial}{\partial s} \frac{s^2}{2} D_r^{(2)} + \frac{\partial}{\partial \phi} s D_\phi^{(1)} - 2 \frac{\partial^2}{\partial s \partial \phi} D_{r\phi}^{(0)} \right) u_{0m''})(v_{0m''}| \rho_{rel.}(t - t'))$$

$$= (v_{1m'}| \left(\frac{\partial}{\partial s} \frac{s^2}{2} D_r^{(2)} + \frac{\partial}{\partial \phi} s D_\phi^{(1)} - 2 \frac{\partial^2}{\partial s \partial \phi} D_{r\phi}^{(0)} \right) \frac{c(\phi)}{\sqrt{\pi}} \exp(-c^2 s^2) P(\phi, t - t'))$$

$$
\begin{aligned}
= & \iint Z_1 \exp(B(\phi'))\exp\left(2\pi i m \frac{G(\phi')}{G(2\pi)}\right) 2c(\phi')\left(-\frac{s'^2}{2}D_r^{(2)}(\phi') + \frac{\partial}{\partial\phi'}s'^2 D_\phi^{(1)}(\phi')\right.\\
& \left. +2\frac{\partial}{\partial\phi'}D_{r\phi}^{(0)}(\phi')\right)\frac{c(\phi')}{\sqrt{\pi}}\exp(-c^2(\phi')s'^2)P(\phi', t-t')ds'd\phi'\\
= & Z_1 \int \exp(B(\phi'))\exp\left(2\pi i m\frac{G(\phi')}{G(2\pi)}\right)\left(-\frac{D_r^{(2)}(\phi')}{2c(\phi')} + c(\phi')\frac{\partial}{\partial\phi'}\frac{D_\phi^{(1)}(\phi')}{c^2(\phi')}\right.\\
& \left. +4c(\phi')\frac{\partial}{\partial\phi'}D_{r\phi}^{(0)}(\phi')\right)P(\phi', t-t')d\phi' \quad . \qquad (59)
\end{aligned}
$$

Inserting eq.(58), (59) and (24_2) into eq.(57) one obtains

$$
\begin{aligned}
& \mathbf{P\Lambda_1 Q}e^{-\mathbf{\Lambda_0}t'}\mathbf{Q\Lambda_1 P}\rho_{rel.}(t-t')\\
= & \frac{c(\phi)}{\sqrt{\pi}}\exp(-c^2s^2)\frac{\partial}{\partial\phi}\frac{D_\phi^{(1)}}{c(\phi)}\int K(t', \phi, \phi')\exp(B(\phi') - B(\phi))\left(-\frac{D_r^{(2)}(\phi')}{4c(\phi')}\right.\\
& \left. +2c(\phi')\frac{\partial}{\partial\phi'}\left(D_{r\phi}^{(0)}(\phi') + \frac{D_\phi^{(1)}(\phi')}{4c^2(\phi')})\right)\right)P(\phi', t-t')d\phi' \qquad (60)
\end{aligned}
$$

where the definition of the kernel (33) and of the 2π periodic continuation of the $\delta-$ function

$$
\delta_{2\pi}(x) := \sum_{n=-\infty}^{\infty}\delta(x - n2\pi) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\exp(-imx) \qquad (61)
$$

has been used.

Eq.(60) and (56) together with eq.(31) yield the result (32) if an integration with respect to s is performed.

Appendix D

Inserting the ansatz (39_2) into eq.(36) and equating powers in ϵ^2 one obtains a hierarchy of equations

$$
\frac{\partial P_0}{\partial\tau_0} + \mathbf{\Lambda_0}P_0 \quad = \quad 0
$$

$$\begin{aligned}\frac{\partial P_1}{\partial \tau_0} + \mathbf{\Lambda_0} P_1 &= -\left(\frac{\partial P_0}{\partial \tau_1} + \mathbf{\Lambda_1} P_0\right) \\ &\vdots\end{aligned} \tag{62}$$

where the abbreviations $\tau_i := \epsilon^{2i} t$ have been introduced. An expansion of the solution of the first equation into eigenfunctions of $\mathbf{\Lambda_0}$ (cf. eq.(41)) leads to

$$P_0(\tau_0, \tau_1, \ldots) = \sum_m a_m(\tau_1, \tau_2, \ldots) \exp(-\lambda_m^{(0)} \tau_0) u_m^{(0)} \quad . \tag{63}$$

The unknown coefficients a_m are determined by the additional equations of the hierarchy. Multiplying eq.(62_2) with the solution of the hermitian conjugate problem that means with $\exp(\lambda_n^{(0)}\tau_0)(v_n^{(0)}|$ one obtains

$$\begin{aligned}\frac{\partial}{\partial \tau_0} \exp(\lambda_n^{(0)}\tau_0)(v_n^{(0)}|P_1) &= -\sum_m \exp((\lambda_n^{(0)} - \lambda_m^{(0)})\tau_0) \\ &\quad \cdot \left(\frac{\partial a_m}{\partial \tau_1}\delta_{nm} + (v_n^{(0)}|\mathbf{\Lambda_1} u_m^{(0)}) a_m\right) \quad .\end{aligned} \tag{64}$$

This equation can be integrated directly. To avoid the secular term emerging in the sum in the case $m = n$ one has to demand

$$\frac{\partial a_n}{\partial \tau_1} + (v_n^{(0)}|\mathbf{\Lambda_1} u_n^{(0)}) a_n = 0 \quad . \tag{65}$$

Taking eq.(41) into account the solution reads

$$a_n(\tau_1, \tau_2, \ldots) = b_n(\tau_2, \ldots) \exp(-\lambda_n^{(1)} \tau_1) \quad . \tag{66}$$

With eq.(63) and neglecting the slow time dependence of higher orders one obviously recovers the approximation (40).

Figure captions

Fig.1: Diagramatic view of the distribution function (2).

Fig.2: Stationary solution of the stochastic van der Pol oscillator for
a) $\epsilon = 5$, b) $\epsilon = 1$.
i) deterministic limit cycle
ii) reduced distribution function $P(\phi)$ (not normalized)
iii) radial fluctuations

Fig.3: Time dependent solutions for $\epsilon^2 D = 0.0005$ and $A = 1.02$ ( ), $A = 1.2$ ( ) respectively.

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